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SPECIAL PSEUDO LINEAR  
ALGEBRAS USING  $[0, n)$

# Special Pseudo Linear Algebras using $[0,n)$

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## PREFACE

In this book we introduce some special type of linear algebras called pseudo special linear algebras using the interval  $[0, n)$ . These new types of special pseudo interval linear algebras has several interesting properties. Special pseudo interval linear algebras are built over the subfields in  $Z_n$  where  $Z_n$  is a S-ring. We study the substructures of them.

The notion of Smarandache special interval pseudo linear algebras and Smarandache strong special pseudo interval linear algebras are introduced. The former S-special interval pseudo linear algebras are built over the S-ring itself. Study in this direction has yielded several interesting results.

S-strong special pseudo interval linear algebras are built over the S-pseudo interval special ring  $[0, n)$ . SSS-pseudo special linear algebras are mainly introduced for only on these new structures, study, develop, describe and define the notion of SSS-linear functionals, SSS-eigen values, SSS-eigen vectors and SSS-polynomials.

This type of study is important and interesting. Authors are sure these structures will find applications as in case of usual linear algebras.

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## Chapter One

# INTRODUCTION

In this chapter we recall the operations on the special interval  $[0, n)$ ,  $n < \infty$  where addition and multiplication are performed modulo  $n$ . This is a special study for  $Z_n \subseteq [0, n)$  and  $[0, n)$  can be realized as the real closure of  $Z_n$ .

$[0, n)$  is never a group under product only a semigroup. Thus if  $n$  is a prime  $[0, n)$  happens to be an infinite pseudo integral domain.

Three types of vector spaces are constructed using  $[0, n)$ . This study is new and innovative. For always  $Z_n$  is imagined to be a ring (a field in case  $n$  is a prime) with only  $n$  number of elements in them. However  $[0, n)$  has infinite number of elements in them.

When  $[0, n)$  is used ( $n$  a prime) then we have usual vector spaces constructed using  $[0, n)$  over the field  $Z_p$ .

The next stage of study being  $S$ -special interval vector spaces using  $[0, n)$  over the  $S$ -ring  $Z_n$ . This will have meaning only if  $Z_n$  is a Smarandache ring. Finally we define the new



notion of Smarandache Strong Special pseudo vector space (SSS-pseudo vector space) over the S-special pseudo interval ring  $[0, n)$ .

Only these happen to pave way for finite dimensional vector space using  $[0, n)$ . Further only using this structure one can define the notion of SSS-linear functionals and the SSS-dual space.

Such type of study is carried out for the first time.

This study will certainly lead to several new algebraic structure inventions.

## Chapter Two

# SPECIAL PSEUDO LINEAR ALGEBRAS USING THE INTERVAL $[0, n)$

In this chapter authors for the first time define special interval vector spaces defined over the S-rings. It is important to keep on record that  $[0, n)$  can maximum be a pseudo integral domain in case  $n$  is a prime and is Smarandache pseudo special interval ring whenever  $Z_n \subseteq [0, n)$  is a S-ring.

Thus all special vector spaces built using  $[0, n)$  and special vector linear algebras built using  $[0, n)$  are only S-linear algebras unless we make, these structures over fields contained in  $[0, n)$ . We will develop and describe them in this chapter.

**DEFINITION 2.1:** *Let  $V = ([0, n), +)$  be an additive abelian group.  $F \subseteq Z_n$  be a field so that  $Z_n$  is a Smarandache ring if  $n$  is a prime  $F = Z_p$  is a field; we define  $V$  to be the special interval vector space over the field  $F \subseteq Z_n \subseteq [0, n)$ . Further we do not demand  $(a + b)v = av + bv$  and  $a(v_1 + v_2) = av_1 + av_2$  for  $a, b \in F \subseteq Z_n$  and  $v, v_1, v_2 \in V$ . The only criteria is  $av = va \in V$  for all  $a \in F \subseteq Z_n$  and  $v \in V$ .*

**Note:**  $V$  can be an additive abelian group built using  $[0, n)$  that is the only criteria for the construction of special interval vector spaces. The distributive laws may or may not be true.

**Example 2.1:** Let  $V = \{[0, 5), +\}$  be the special interval vector space over  $Z_5$ ; the field of modulo integers.

**Example 2.2:** Let  $V = \{[0, 6), +\}$  be a special interval vector space over the field  $F = \{0, 2, 4\} \subseteq Z_6 \subseteq [0, 6)$  or over the field  $F_1 = \{0, 3\} \subseteq [0, 6)$ .

**Example 2.3:** Let  $V = \{[0, 15), +\}$  be the special interval vector space over the field  $F = \{0, 3, 6, 9, 12\} \subseteq Z_{15}$ .

Let  $0.315 \in V$  then for  $a = 6$  we have  $a \times 0.315 = 1.890 \in V$ .

Let  $a = 3$  and  $b = 9 \in V$ . For  $v = 6.021 \in V$  we have

$$\begin{aligned} (a + b) v &= (3 + 9)v \\ &= 12 \times 6.021 = 72.252 \\ &= 12.252 \end{aligned} \quad \dots (1)$$

$$\begin{aligned} av + bv &= 3 \times 6.021 + 9 \times 6.021 \\ &= 18.063 + 54.189 \\ &= 72.252 \\ &= 12.252 \end{aligned} \quad \dots (2)$$

(1) and (2) are identical in this case hence  $(a + b) v = av + bv$  in  $V$  as a special interval vector space over  $F$ .

Let  $v_1 = 2.615$  and  $v_2 = 7.215 \in V$  and  $a = 6 \in F$ .

We find

$$\begin{aligned} a(v_1 + v_2) &= 6(2.615 + 7.215) = 6(9.830) = 58.980 \\ &= 13.980 \end{aligned} \quad \dots \text{ I}$$

$$\begin{aligned} av_1 + av_2 &= 6 \times 2.615 + 6 \times 7.215 \\ &= 15.690 + 43.290 \\ &= 58.980 \\ &= 13.980 \end{aligned} \quad \dots \text{ II}$$

I and II are identical in this case hence this set of vectors in  $V$  distribute over the scalar, however one do not demand this condition in our definition.

$0.v = 0$  for all  $v \in V$  and  $a.0 = 0$  for all  $a \in F$ .

The following observations are important

1. Always the cardinality of  $V$  is infinite.
2. The advantage when  $n$  is a composite number and if  $Z_n$  is  $S$ -ring we can have more than one special interval vector space.

The number of spaces depends on the number of fields the ring  $Z_n$  has.

**Example 2.4:** Let  $V = \{[0, 12), +\}$  be a special interval vector space over the field  $F_1 = \{0, 4, 8\}$ . We see this special interval vector space can be defined only over one field. For  $Z_{12}$  has only one subset which is a field.

**Example 2.5:** Let  $V = \{[0, 24), +\}$  be a special interval vector space over the field  $F_1 = \{0, 8, 16\} \subseteq Z_{24}$ . This is only special interval vector space of the interval  $[0, 24)$  over the field in  $Z_{24}$ .

**Example 2.6:** Let  $V_1 = \{[0, 30), +\}$  be a special interval vector space over the field  $F_1 = \{0, 15\} \subseteq Z_{30}$ .

$V_2 = \{[0, 30), +\}$  be a special interval vector space over the field  $F_2 = \{0, 10, 20\} \subseteq Z_{30}$ .

$V_3 = \{[0, 30), +\}$  be the special interval vector space over the field  $F_3 = \{0, 6, 12, 18, 24\} \subseteq [0, 30)$ ; we have only three vector spaces over the three fields in  $Z_{30} \subseteq [0, 30)$ .

**Example 2.7:** Let  $V = \{[0, 23), +\}$  be a special interval vector space over the field  $Z_{23}$ . We have a unique special interval vector space over the field  $F = Z_{23}$ .

**Example 2.8:** Let  $V = \{[0, 29), +\}$  be a special interval vector space over the field  $Z_{29} = F$ .

**Example 2.9:** Let  $V_1 = \{[0, 143), +\}$  be a special interval vector space over the field  $Z_{143} = F$ .

**Example 2.10:** Let  $V = \{[0, 43), +\}$  be a special interval vector space over the field  $F = Z_{43}$ .

In view of all this we have the following theorems.

**THEOREM 2.1:** *Let  $V = \{[0, p), +\}$  be the special interval vector space over the field  $Z_p = F$  ( $p$  a prime);  $V$  is the only one special interval vector space over  $Z_p$ .*

Proof is direct hence left as an exercise to the reader.

**THEOREM 2.2:** *Let  $V = \{[0, n), +\}$  be the special interval vector space over  $t$  fields,  $F_i \subseteq Z_n$ ,  $1 \leq i \leq t$ , hence using this  $V$  we have  $t$  distinct special interval vector spaces over each of the fields,  $F_i$ , ( $1 \leq i \leq t$ ).*

**Proof :** If  $Z_n$  is a S-ring and  $Z_n$  has  $t$  number of subsets  $F_i$  such that each  $F_i$  is a field then we have  $V = \{[0, n), +\}$  to be special interval vector space over  $F_i$  for  $i = 1, 2, \dots, t$ .

Hence the claim.

We will illustrate this situation by some examples.

**Example 2.11:** Let  $V = \{[0, 42), +\}$  be a special interval vector space over the field  $F_1 = \{0, 21\} \subseteq Z_{42}$ .

Let  $F_2 = \{0, 14, 28\} \subseteq Z_{42}$  be a field in  $Z_{42}$ .

$V_2 = \{[0, 42), +\}$  is a special interval vector space over the field  $F_2$ .

Let  $F_3 = \{0, 6, 12, 18, 24, 30, 36\} \subseteq Z_{42}$  be the field.  $(F_3 \setminus \{0\}, \times)$  is given by the following table

$\times$	6	12	18	24	30	36
6	36	30	24	18	12	6
12	30	18	6	36	24	12
18	24	6	30	12	36	18
24	18	36	12	36	6	24
30	12	24	36	6	18	30
36	6	12	18	24	30	36

36 acts as the identity with respect to multiplication of the field  $F_3 = \{0, 6, 12, 18, 24, 30, 36\} \subseteq Z_{42}$ .

We have three different special interval vector spaces.

For if  $0.65 \in V$  now  $V$  as a special interval vector space over  $F_1$  we get

$$0.65 \times 21 = 13.65 \in V.$$

$$\text{Now } 0.65 \times 14 = 9.10 \in V \text{ and } 0.65 \times 30 = 19.50 \in V.$$

We see the three spaces are distinct.

Now we proceed onto discuss about special interval vector subspaces of a special interval vector spaces.

**Example 2.12:** Let  $V = \{[0, 15), +\}$  be the special interval vector space over the field  $F = \{0, 5, 10\} \subseteq Z_{15}$ .

$P_1 = \{0, 1, 2, 3, \dots, 14\} \subseteq V$  is a special interval vector subspace of  $V$  over  $F$ .

$P_2 = \{0, 3, 6, 9, 12\} \subseteq V$  is also a special vector subspace of  $V$  over  $F$ .

**Example 2.13:** Let  $V = \{[0, 7), +\}$  be a special interval vector space over the field  $F = \mathbb{Z}_7 = \{0, 1, 2, \dots, 6\}$ .

$P = \{0, 1, 2, 3, 4, 5, 6\} \subseteq V$  is a subspace of  $V$  over  $F = \mathbb{Z}_7$ .

$P_1 = \{0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, \dots, 6.5\} \subseteq V$  is a special interval vector subspace of  $V$  over  $\mathbb{Z}_7$ .

**Example 2.14:** Let  $V = \{[0, 3), +\}$  be a special interval vector space over the field  $F = \{0, 1, 2\} = \mathbb{Z}_3$ .  $P_1 = \{0, 1, 2\} \subseteq V$  is a subspace of  $V$  over  $F$ .  $P_2 = \{0, 0.5, 1, 1.5, 2, 2.5\} \subseteq V$  is again a subspace of  $V$  over  $F$ .

$P_3 = \{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, 1.6, 1.8, 2, 2.2, 2.4, 2.6, 2.8\} \subseteq V$  is also a subspace of  $V$  over  $F$ .

$P_4 = \{0, 0.25, 0.5, 0.752, 1, 1.25, 1.50, 1.75, 2, 2.25, 2.5, 2.75\} \subseteq V$  is also a subspace of  $V$  over  $F$ .

**Example 2.15:** Let  $V = \{[0, 12), +\}$  be the special interval vector space over the field  $F = \{0, 4, 8\}$ .  $P_1 = \{0, 1, 2, \dots, 10, 11\} \subseteq V$  is a special interval vector subspace of  $V$  over  $F$ .

$P_2 = \{0, 6\} \subseteq V$  is a special interval vector subspace of  $V$  over  $F$ .

$P_3 = \{0, 3, 6, 9\} \subseteq V$  is a special interval vector subspace of  $V$  over  $F$ .

$P_4 = \{0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, \dots, 10, 10.5, 11, 11.5\} \subseteq V$  is a special interval vector subspace of  $V$  over  $F$ .

$P_5 = \{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, \dots, 1.8, 2, 2.2, 2.4, \dots, 10, 10.2, 10.4, 10.6, 10.8, 11, 11.2, 11.4, 11.6, 11.8\} \subseteq V$  is a special interval vector subspace of  $V$  over  $F$ .

**Example 2.16:** Let  $V = \{[0, 10), +\}$  be the special interval vector space over the field  $F = \{0, 2, 4, 6, 8\} \subseteq \mathbb{Z}_{10}$ .

$P_1 = \{0, 5\} \subseteq V$  is a subspace of  $V$  over  $F$ .

$P_2 = \{0, 1, 2, \dots, 9\} \subseteq V$  is a subspace of  $V$  over  $F$ .

$P_3 = \{0, 0.5, 1, 1.5, 2, 2.5, \dots, 9, 9.5\} \subseteq V$  is a subspace of  $V$  over  $F$ .

$P_4 = \{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, \dots, 9, 9.2, 9.4, 9.6, 9.8\} \subseteq V$  is again a subspace of  $V$  over  $F$ .

$Z_{10}$  has only two subsets which are fields; viz  $F = \{0, 5\}$  and  $F_1 = \{0, 2, 4, 6, 8\}$ .

**Example 2.17:** Let  $V = \{[0, 60), +\}$  be a special interval vector space over the field  $F = \{0, 20, 40\} \subseteq Z_{60}$ .

$P_1 = \{0, 1, 2, \dots, 59\} \subseteq V$  is a special interval vector subspace of  $V$  over  $F$ .

$P_2 = \{0, 2, 4, 6, 8, \dots, 58\} \subseteq V$  is again a special interval subspace of  $V$  over  $F$ .

$P_3 = \{0, 3, 6, 9, 12, \dots, 57\} \subseteq V$  is also special interval subspace of  $V$  over  $F$ .

$P_4 = \{0, 6, 12, 18, 24, 30, \dots, 54\} \subseteq V$  is again a special interval subspace of  $V$  over  $F$ .

$P_5 = \{0, 10, 20, 30, 40, 50\} \subseteq V$  is a special interval subspace of  $V$  over  $F$ .

$P_6 = \{0, 15, 30, 45\} \subseteq V$  is also a special interval subspace of  $V$  over  $F$ .

$P_7 = \{0, 12, 24, 36, 48\} \subseteq Z_{60}$  is again a special interval subspace of  $V$  over  $F$ .

$P_8 = \{0.5, 0, 1, 1.5, 2, 2.5, \dots, 58.5, 59, 59.5\} \subseteq Z_{60}$  is also a subspace of  $V$  over  $F$ .



In fact we chose  $F_1 = \{0, 12, 24, 36, 48\}$  as a field then also we can get subspaces of that  $V$  over  $F_1$ .

**Example 2.18:** Let  $V = \{[0, 73), +\}$  be a special interval vector space over the field  $Z_{73}$ . This  $V$  has subspaces given by  $M_1 = \{0, 1, 2, \dots, 73\} \subseteq V$  is a subspace of  $V$  over  $Z_{73}$ .

$M_2 = \{0, 0.5, 1, 1.5, 2, 2.5, \dots, 72, 72.5\} \subseteq V$  is again a subspace of  $V$  over  $Z_{73}$ .

$M_3 = \{0, 0.125, 0.250, 0.375, 0.5, 0.625, \dots, 72.125, 72.250, 72.375, 72.5, 72.625, 72.750, 72.875\} \subseteq V$  is again a subspace of  $V$  over  $Z_{73}$ . Even if in  $[0, p)$ ,  $p$  is a prime we see all special interval spaces have subspaces.

**THEOREM 2.3:** Let  $V = \{[0, p), +\}$  be a special interval vector space over the field  $Z_p$  ( $p$  a prime);  $V$  has several subspaces.

Proof is direct hence left as an exercise to the reader.

**Corollary :** If  $[0, p)$  in theorem 2.3  $p$  is replaced by  $n$ ;  $n$  a composite number still  $V = \{[0, n), +\}$  has several subspaces.

It is important and interesting to note all special interval vector spaces  $V$  defined using the interval  $[0, n)$  is such that  $|V| = \infty$ .

We will proceed onto define the concept of linear dependence and linear independence and a basis of  $V$  over the field  $F \subseteq [0, n)$ .

Let  $V = \{[0, 27), +\}$  be a special vector space over the field  $F = \{0, 9, 18\} \subseteq [0, 27)$ .

Take  $x = 2.04$  and  $y = 3.3313 \in V$  we see  $2.04$  and  $3.3313$  are linearly independent for this  $x$  and  $y$  are not related by any scalar from  $F$ .

Let  $x = 2.01$  and  $y = 18.09 \in V$  we see  $y = 9x$  so  $y$  and  $x$  are linearly dependent in  $V$  over the field  $F$ .

We can have several such concepts. We say a set of elements  $B = \{v_1, v_2, \dots, v_n\} \in V$  is a linearly independent set in  $V$ , if no  $v_i$  can be expressed in terms of  $v_j$ 's  $i \neq j, j, i = 1, 2, \dots, n$ . That is  $v_i \neq \sum \alpha_i v_j$  where  $\alpha_i$ 's  $\in F$  and  $v_j \in B$ .

We say the linearly independent set  $B$  is said to be a basis if  $B$  generates  $V$ .

Recall here also we say

$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$  is possible if and only if each  $\alpha_i = 0$ . The authors feel that  $V$  has only infinite subset of  $V$  to be a basis.

That is  $V$  cannot have a finite basis over the field  $F \subseteq [0, n)$ .

**Example 2.19:** Let  $V = \{[0, 5), +\}$  be a vector space over the field  $F = \{0, 1, 2, 3, 4\} = Z_5 \subseteq [0, 5)$ .  $V$  is an infinite dimensional vector space over  $F$ .

$V$  cannot have a finite basis.

However if  $P_1 = \{0, 1, 2, 3, 4\} \subseteq [0, 5) \subseteq V$  be a special interval vector subspace of  $V$  then  $P_1$  has dimension 1 over  $F$ .  $\{1\}$  or  $\{2\}$  or  $\{3\}$  or  $\{4\}$  is a basis of  $P_1$  over  $F$ .

Let  $P_2 = \{0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5\}$  be vector subspace of  $V$  over  $F$ .

$B = \{0.5\} \subseteq P_2$  is a basis of  $V$  over  $F$ . Thus dimension of  $P_2$  over  $F$  is one.

**Example 2.20:** Let  $V = \{[0, 13), +\}$  be a special interval vector space over the field  $Z_{13}$ .

$V$  is an infinite dimensional vector space over  $Z_{13}$ .

$V$  has also subspaces which are finite dimensional.

Next we proceed onto study the special interval vector spaces built using the interval group  $G = \{[0, n), +\}$ .

**Example 2.21:** Let  $V = \{(a_1, a_2, a_3) \mid a_i \in [0, 15); 1 \leq i \leq 3, +\}$  be the special interval vector space over the field  $F = \{0, 5, 10\} \subseteq \mathbb{Z}_{15}$ .  $V$  has infinite special interval vector subspaces also.

For take  $P_1 = \{(a_1, 0, 0) \mid a_1 \in [0, 15), +\} \subseteq V$ ,

$P_2 = \{(0, a_1, 0) \mid a_1 \in [0, 15), +\} \subseteq V$ ,

$P_3 = \{(0, 0, a_1) \mid a_1 \in [0, 15), +\} \subseteq V$ ,

$P_4 = \{(0, a_1, a_2) \mid a_1, a_2 \in [0, 15), +\} \subseteq V$ ,

$P_5 = \{(a_1, a_2, 0) \mid a_1, a_2 \in [0, 15), +\} \subseteq V$  and

$P_6 = \{(a_1, 0, a_2) \mid a_1, a_2 \in [0, 15), +\} \subseteq V$  are the six vector subspaces of  $V$  and  $|P_i| = \infty$ ;  $1 \leq i \leq 6$  and all of them are infinite dimensional over the field  $F = \{0, 5, 10\} \subseteq \mathbb{Z}_{15}$ .

We have subspace  $M_i$  such that  $|M_i| < \infty$ . For take  $M_1 = \{(a_1, 0, 0) \mid a_1 \in \{0, 5, 10\}, +\} \subseteq V$  is a subspace of  $V$  of dimension 1 over  $F$ .

$M_2 = \{(0, a_1, 0) \mid a_1 \in \{0, 3, 6, 9, 12\} \subseteq [0, 15)\} \subseteq V$  is also a subspace of  $V$  and  $|M_2| < \infty$ .

We can have atleast 14 such subspaces which has only finite number of elements in them.

**Example 2.22:** Let

$V = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in [0, 23), 1 \leq i \leq 5, +\}$  be the special interval vector space over the field  $F = \mathbb{Z}_{23}$ .

$V$  has both subspaces of finite dimension as well as infinite dimension.

$P_1 = \{(a_1, 0, 0, 0, 0) \mid a_1 \in [0, 23), +\} \subseteq V$  is a subspace of  $V$  of infinite dimension and  $|P_1| = \infty$ .

$P_2 = (a_1, 0, 0, 0, a_2) \mid a_1, a_2 \in [0, 23), +\} \subseteq V$  is also subspace of  $V$  over  $F = Z_{23}$  and of infinite dimension.

$P_3 = (a_1, a_2, 0, 0, 0) \mid a_1, a_2 \in [0, 23), +\} \subseteq V$ ,  
 $P_4 = (0, 0, a_1, a_2, a_3) \mid a_1, a_2, a_3 \in [0, 23), +\} \subseteq V$ ,  
 $P_5 = (0, 0, 0, a_1, 0) \mid a_1 \in [0, 23), +\} \subseteq V$ ,  
 $P_6 = (a_1, a_2, a_3, a_4, 0) \mid a_i \in [0, 23), 1 \leq i \leq 4, +\} \subseteq V$ ,  
 $P_7 = (0, a_1, a_2, a_3, a_4) \mid a_i \in [0, 23), 1 \leq i \leq 4, +\} \subseteq V$ ,  
 $P_8 = (a_1, a_2, 0, a_3, a_4) \mid a_i \in [0, 23), 1 \leq i \leq 4, +\} \subseteq V$  and  
 $P_9 = (0, 0, a_1, a_2, 0) \mid a_i \in [0, 23), 1 \leq i \leq 2, +\}$  are all some of the subspaces of  $V$  over  $Z_{23}$ .

$M_1 = \{(a_1, 0, 0, 0, 0) \mid a_1 \in Z_{23}, +\} \subseteq V$ ,  
 $M_2 = \{(a_1, a_2, 0, 0, 0) \mid a_1, a_2 \in Z_{23}, +\} \subseteq V$ ,  
 $M_3 = \{(0, a_1, 0, 0, 0) \mid a_1 \in Z_{23}, +\} \subseteq V$ ,  
 $M_4 = \{(0, 0, a_1, a_2, 0) \mid a_1, a_2 \in Z_{23}, +\} \subseteq V$  and  
 $M_5 = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in Z_{23}, +, 1 \leq i \leq 5\} \subseteq V$  are some of the subspaces of  $V$  over  $Z_{23}$ .

We see each  $M_i$  is such that  $|M_i| < \infty$  and  $M_i$ 's are finite dimensional over  $Z_{23}$ . For  $1 \leq i \leq 3$ .

Infact we have  ${}_5C_1 + {}_5C_2 + {}_5C_3 + {}_5C_4 + 1$  number of subspace of finite order and finite dimensional over  $F = Z_{23}$ .

**Example 2.23:** Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} \mid a_i \in [0, 24), 1 \leq i \leq 7, + \right\}$$

be a special interval vector space over the field  $F = \{0, 8, 16\} \subseteq \mathbb{Z}_{24}$ .

$V$  has both finite and infinite dimensional vector subspaces.

Let

$$P_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_1 \in [0, 24), + \right\} \subseteq V$$

be a special interval vector subspace over the  $F = \{0, 8, 16\} \subseteq \mathbb{Z}_{24}$ .

Clearly  $P_1$  is infinite dimensional vector subspace of  $V$  over  $F$ .

Consider

$$M_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} \mid a_i \in \{0, 8, 16\}, 1 \leq i \leq 7, + \right\} \subseteq V$$

is a special interval vector subspace of  $V$  over  $F$ .

Infact  $M_1$  is finite dimensional over  $F$  and  $|M_1| < \infty$ .

$$M_2 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_3 \end{bmatrix} \mid a_1, a_2, a_3 \in \{0, 8, 16\}, + \} \subseteq V$$

is a special interval vector subspace of  $V$  over  $F$ .  $M_2$  is finite dimensional over  $F$  and  $|M_2| < \infty$ .

$$M_3 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ a_2 \\ 0 \\ a_3 \\ 0 \\ a_4 \end{bmatrix} \mid a_i \in \{0, 8, 16\}, 1 \leq i \leq 4, + \} \subseteq V$$

is a special interval vector subspace of  $V$  over  $F$ .

$M_3$  is also finite dimensional over  $F$  and  $|M_3| < \infty$ .

$$M_4 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \mid a_i \in \{0, 8, 16\}, 1 \leq i \leq 3, + \} \subseteq V$$

is a special interval vector subspace of  $V$  over  $F$ .

Clearly  $M_4$  is also finite dimensional over  $F$ .

Let

$$B_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_1, a_2 \in [0, 24), + \right\} \subseteq V$$

be the special interval vector subspace of  $V$  over  $F$ .

$B_1$  is infinite dimensional subspace of  $V$  over  $F$ .

Let

$$B_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \mid a_1, a_2, a_3 \in [0, 24), + \right\} \subseteq V$$

be the special interval vector subspace of  $V$  over  $F$ .

Clearly  $B_2$  is infinite dimensional over  $F$ .

$V$  has several subspaces which are infinite dimensional over  $F$ .

In fact if

$$T_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_1 \in [0, 24), + \} \subseteq V,$$

$$T_2 = \left\{ \begin{bmatrix} 0 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_2 \in [0, 24), + \} \subseteq V,$$

$$T_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ a_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_3 \in [0, 24), + \} \subseteq V,$$



$$T_4 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_4 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_4 \in [0, 24), + \} \subseteq V,$$

$$T_5 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_5 \\ 0 \\ 0 \end{bmatrix} \mid a_5 \in [0, 24), + \} \subseteq V,$$

$$T_6 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_6 \\ 0 \end{bmatrix} \mid a_6 \in [0, 24), + \} \subseteq V \text{ and}$$

$$T_7 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_7 \end{bmatrix} \mid a_7 \in [0, 24), + \right\} \subseteq V$$

be seven distinct special interval vector subspaces of  $V$  over  $F$ .

Clearly

$$T_i \cap T_j = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}, i \neq j, 1 \leq i, j \leq 7 \text{ and}$$

$T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 = V$  is the distinct sum of subspaces of  $V$  over  $F$ .

Each subspace is infinite dimensional over  $F$ .

Let

$$W = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} \mid a_i \in \{0, 8, 16\} \subseteq \mathbb{Z}_{24}, 1 \leq i \leq 7 \right\} \subseteq V$$

be a special interval vector subspace of  $V$  of dimension seven over  $F$ .

$$\text{Let } S_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_1 \in \{0, 8, 16\} \subseteq \mathbb{Z}_{24}, + \right\} \subseteq V,$$

$$S_2 = \left\{ \begin{bmatrix} 0 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_2 \in \{0, 8, 16\} \subseteq \mathbb{Z}_{24}, + \right\} \subseteq V,$$

$$S_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ a_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_3 \in \{0, 8, 16\} \subseteq \mathbb{Z}_{24}, + \} \subseteq V,$$

$$S_4 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_4 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_4 \in \{0, 8, 16\} \subseteq \mathbb{Z}_{24}, + \} \subseteq V,$$

$$S_5 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_5 \\ 0 \\ 0 \end{bmatrix} \mid a_5 \in \{0, 8, 16\} \subseteq \mathbb{Z}_{24}, + \} \subseteq V,$$

$$S_6 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_6 \\ 0 \end{bmatrix} \mid a_6 \in \{0, 8, 16\} \subseteq \mathbb{Z}_{24}, + \} \subseteq V \text{ and}$$

$$S_7 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_7 \end{bmatrix} \mid a_7 \in \{0, 8, 16\} \subseteq \mathbb{Z}_{24}, + \} \subseteq V$$

be seven different subspaces of  $V$  over  $F$ .

Each of them is also a subspace of  $W \subseteq V$  over  $F$ .

Clearly

$$S_i \cap S_j = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}, i \neq j, 1 \leq i, j \leq 7 \text{ and}$$

$S_1 + S_2 + S_3 + S_4 + S_5 + S_6 + S_7 \subseteq V$  is not a direct sum of  $V$  but is certainly a direct sum of  $W$ .

Thus we call such sum as subdirect sum of subspaces of  $W$ .

In fact  $W$  can be written as direct sum in other ways also.

Let

$$P_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_1, a_2 \in \{0, 8, 16\} \subseteq \mathbb{Z}_{24}, + \} \subseteq W \subseteq V,$$

$$P_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_1, a_2 \in \{0, 8, 16\} \subseteq \mathbb{Z}_{24}, + \} \subseteq V \text{ and}$$

$$P_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \mid a_1, a_2, a_3 \in \{0, 8, 16\} \subseteq \mathbb{Z}_{24}, + \} \subseteq W \subseteq V$$

be special interval vector subspace of  $W$  as well as  $V$ .

We see

$$P_i \cap P_j = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}, i \neq j, 1 \leq i, j \leq 3$$

and  $W = P_1 + P_2 + P_3 \subseteq V$ ; so  $P_i$ 's give a subsubdirect of subsubspaces of  $V$ .

Infact the representation of  $V$  (or  $W \subseteq V$ ) as a direct sum of sub subdirect subsubspace sum is not unique as in case of usual spaces.

**Example 2.24:** Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \mid a_i \in [0, 23), 1 \leq i \leq 6, + \right\}$$

be the special interval column matrix vector space over the field  $Z_{23}$ .

$V$  has both finite and infinite subspaces. Infact we can have subspaces  $W$  in  $V$  such that there exists  $W^\perp$  in  $V$  such that  $W \oplus W^\perp = V$ .

Also we have subspaces  $T$  in  $V$  such that  $T^\perp$  of  $T$  is only a proper subset of  $V$ .

First we will illustrate both these in this  $V$ .

$$W = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_i \in [0, 23), 1 \leq i \leq 3, +\} \subseteq V$$

is a subspace of  $V$  over  $Z_{23}$ .

We see

$$W^\perp = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \mid a_i \in [0, 23), 1 \leq i \leq 3, +\} \subseteq V$$

is a subspace of  $V$  and

$$W \cap W^\perp = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ and } W + W^\perp = V.$$



Now take

$$S_1 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_1 \end{bmatrix} \mid a_1 \in [0, 23), + \right\} \subseteq V$$

is a subspace of  $V$  and

$$S_1 \cap W = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ but } S + W \neq V$$

we see  $S$  is orthogonal with  $W$  but  $S_1 + W \neq V$ .

Similarly

$$S_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ 0 \end{bmatrix} \mid a_1 \in [0, 23), + \right\} \subseteq V$$

is orthogonal with  $W$  but  $S_1 + W \neq V$  and

$$S_2 \cap W = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Now consider

$$P_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_1, a_2 \in \{0, 1, 2, \dots, 22\} \subseteq [0, 23), + \right\}$$

a vector subspace of  $V$ . Clearly  $P_1$  is finite dimensional subspace of  $V$  over  $Z_{23}$ .

$$B_1 = \left\{ \begin{bmatrix} 0 \\ a_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_1 \in [0, 23), + \right\} \subseteq V$$

is such that

$$P_1 \cap B_1 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ and } P_1 + B_1 \neq V$$

infact  $P_1$  is finite dimensional over  $Z_{23}$ .

$$B_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \mid a_1 \in [0, 23), a_2, a_3 \in Z_{23}, + \right\} \subseteq V$$

is again a subspace of  $V$  of infinite dimensional over  $F = Z_{23}$ .

$$\text{Clearly } B_2 \cap P_1 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ and } B_2 + P_1 \subseteq V.$$

Thus  $V$  has both finite dimensional and infinite dimensional vector subspaces over the field  $F = Z_{23}$ .

**Example 2.25:** Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \mid a_i \in [0, 43), 1 \leq i \leq 9, + \right\}$$

be a special interval vector space over the field  $F = \mathbb{Z}_{43}$ .  $V$  has several subspaces both of finite and infinite dimension. However  $V$  is infinite dimensional over the field  $F = \mathbb{Z}_{43}$ .

Let

$$W_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid a_1 \in [0, 43), + \right\} \subseteq V,$$

$$W_2 = \left\{ \begin{bmatrix} 0 & a_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid a_2 \in [0, 43), + \right\} \subseteq V,$$

$$W_3 = \left\{ \begin{bmatrix} 0 & 0 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid a_3 \in [0, 43), + \right\} \subseteq V,$$

$$W_4 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ a_4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid a_4 \in [0, 43), + \right\} \subseteq V,$$

$$W_5 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid a_5 \in [0, 43), + \right\} \subseteq V,$$

$$W_6 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a_6 \\ 0 & 0 & 0 \end{bmatrix} \mid a_6 \in [0, 43), + \} \subseteq V,$$

$$W_7 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_7 & 0 & 0 \end{bmatrix} \mid a_7 \in [0, 43), + \} \subseteq V,$$

$$W_8 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_8 & 0 \end{bmatrix} \mid a_8 \in [0, 43), + \} \subseteq V \text{ and}$$

$$W_9 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_9 \end{bmatrix} \mid a_9 \in [0, 43), + \} \subseteq V$$

are all vector subspaces of  $V$  of infinite dimension over the field  $F = \mathbb{Z}_{43}$ .

We see

$$W_i \cap W_j = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \text{ if } i \neq j, 1 \leq i, j \leq 9$$

and  $W_1 + W_2 + \dots + W_9 = V$  is a direct sum.

This is the maximum number of subspaces of  $V$  in which  $V$  is written as direct sums can have the number of subspaces to be strictly less than or equal to nine.

Consider

$$R_1 = \left\{ \begin{bmatrix} a_1 & a_2 & 0 \\ 0 & 0 & 0 \\ a_3 & a_4 & 0 \end{bmatrix} \mid a_i \in [0, 43), 1 \leq i \leq 4, + \right\} \subseteq V,$$

$$R_2 = \left\{ \begin{bmatrix} 0 & 0 & a_1 \\ a_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid a_1, a_2 \in [0, 43), 1 \leq i \leq 2, + \right\} \subseteq V \text{ and}$$

$$R_3 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_1 & a_2 \\ 0 & 0 & a_3 \end{bmatrix} \mid a_1, a_2, a_3 \in [0, 43), + \right\} \subseteq V$$

are vector subspaces of  $V$  over the field  $F = Z_{43}$ .

$$R_i \cap R_j = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \text{ if } i \neq j, 1 \leq i, j \leq 3 \text{ and}$$

$R_1 + R_2 + R_3 = V$  is thus a direct sum. Each  $R_i$  is an infinite dimensional vector subspace of  $V$  over  $F$ .

Let

$$T_1 = \left\{ \begin{bmatrix} a_1 & a_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \mid a_1, a_2, a_3 \in Z_{43}, + \right\} \subseteq V,$$

$$T_2 = \left\{ \begin{bmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ 0 & 0 & a_3 \end{bmatrix} \mid a_1, a_2, a_3 \in Z_{43}, + \right\} \subseteq V,$$

$$T_3 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ a_1 & a_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid a_1, a_2 \in \mathbb{Z}_{43}, + \} \subseteq V \text{ and}$$

$$T_4 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1 & a_2 & 0 \end{bmatrix} \mid a_1, a_2 \in \mathbb{Z}_{43}, + \} \subseteq V$$

are subspaces of  $V$  over the field  $F = \mathbb{Z}_{43}$ .

All the four spaces are finite dimensional over  $F = \mathbb{Z}_{43}$  and

$$T_i \cap T_j = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \text{ if } i \neq j \text{ and } 1 \leq i, j \leq 4.$$

Further  $T_1 + T_2 + T_3 + T_4 \subseteq V$ .

Suppose

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \mid a_i \in \mathbb{Z}_{43}, 1 \leq i \leq 9 \right\} \subseteq V,$$

then  $M$  is a finite dimensional vector subspace of  $V$  over  $\mathbb{Z}_{43}$ .

Further  $M = T_1 + T_2 + T_3 + T_4$  and this sum we call as sub  
subdirect sum of subspaces of the subspace  $M$  of  $V$ .

The basis for the space

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \mid a_i \in \mathbb{Z}_{43}, 1 \leq i \leq 9 \right\} \text{ is}$$

$$B = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \right.$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\left. \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \subseteq M$$

is a basis of  $M$  over  $\mathbb{Z}_{43}$  and dimension is 9.

Let

$$D = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \mid a_i \in [0, 43); 1 \leq i \leq 3 \right\} \subseteq V$$

be a vector subspace of  $V$  over  $F$  of infinite dimension

$$E = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \mid a_1, a_2, a_3 \in \mathbb{Z}_{43}; + \right\} \subseteq V$$



is a vector subspace of dimension 3 over  $Z_{43}$ .

Clearly  $E \subseteq D$ ; thus a subspace may contain a subspace of finite dimension.

**Example 2.26:** Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix} \mid a_i \in [0, 47), 1 \leq i \leq 30, + \right\}$$

be the special interval vector space over the field  $F = Z_{47}$ .

$V$  has subspaces of finite and infinite dimension over  $F$ .

$V$  can be written as a direct sum of subspaces.

If  $V = W_1 + \dots + W_n$ ; we see  $n = 30$  is the maximum value for  $n$ .

Further

$$W_i \cap W_j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \text{ if } i \neq j, 1 \leq i, j \leq n.$$

Just  $n$  varies between 2 and 30 that is  $2 \leq n \leq 30$ .

We have several subspaces of  $V$  which in general may not lead to a direct sum.

Let

$$P_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ \vdots & \vdots & \vdots \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \mid a_i \in [0, 47), 1 \leq i \leq 18, + \} \subseteq V;$$

$P_1$  is a special interval vector subspace of  $V$  over the field  $F = \mathbb{Z}_{47}$ .

$$P_2 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \mid a_i \in [0, 47), 1 \leq i \leq 12, + \} \subseteq V;$$

$P_2$  is a special interval vector subspace of  $V$  over the field  $F = \mathbb{Z}_{47}$ .

Clearly

$$P_1 \cap P_2 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \right\} \text{ and } V = P_1 + P_2;$$

thus  $V$  is the direct sum of  $P_1$  and  $P_2$  we see  $P_1$  and  $P_2$  are of infinite dimension over  $F = \mathbb{Z}_{47}$ .

Let

$$S_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \mid a_1 \in [0, 47), + \} \subseteq V$$

be the special interval vector subspace of  $V$  over  $F$  of dimension infinity.

$$S_2 = \left\{ \begin{bmatrix} 0 & a_2 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \mid a_2 \in [0, 47), + \} \subseteq V,$$

$$S_3 = \left\{ \begin{bmatrix} 0 & 0 & a_3 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \mid a_3 \in [0, 47), + \} \subseteq V,$$

$$S_4 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ a_4 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \mid a_4 \in [0, 47), + \} \subseteq V \text{ and so on.}$$

$$S_{12} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & a_{12} \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \mid a_{12} \in [0, 47), + \} \subseteq V \text{ and so on.}$$

$$S_{27} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 0 & a_{27} \\ 0 & 0 & 0 \end{bmatrix} \right\} \quad a_{27} \in [0, 47), + \} \subseteq V,$$

$$S_{28} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ a_{28} & 0 & 0 \end{bmatrix} \right\} \quad a_{28} \in [0, 47), + \} \subseteq V,$$

$$S_{29} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & a_{29} & 0 \end{bmatrix} \right\} \quad a_{29} \in [0, 47), + \} \subseteq V \text{ and}$$

$$S_{30} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 0 & a_{30} \end{bmatrix} \right\} \quad a_{30} \in [0, 47), + \} \subseteq V$$

are 30 subspaces of  $V$  over  $F$  which are distinct and each of them are of infinite dimension.

We see

$$P_i \cap P_j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \quad \text{if } i \neq j, 1 \leq i, j \leq 30$$

and  $V = P_1 + \dots + P_{30}$  is a direct sum of subspaces of  $V$ .

**Example 2.27 :** Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{30} \\ a_{31} & a_{32} & \dots & a_{40} \end{bmatrix} \mid a_i \in [0, 12), 1 \leq i \leq 40, + \right\}$$

be the special interval vector space over  $F = \{0, 4, 8\} \subseteq \mathbb{Z}_{12}$ .

$V$  has several subspaces some of which are infinite dimension and some are of finite dimension.

We can write  $V$  as a direct sum of subspaces of  $V$  over  $F$ .

Thus in view of all these we have the following theorem.

**THEOREM 2.4:** *Let*

$V = \{m \times n \text{ matrices with entries from } [0, n), +\}$  *be a special interval vector space over a field*  $F = \mathbb{Z}_n$ .

- (1)  $V$  *has infinite dimensional and finite dimensional subspaces over*  $F$ .
- (2)  $V = W_1 + \dots + W_t$  *and*  $W_i \cap W_j = (0)$ , *zero matrix if*  $i \neq j$ ,  $1 \leq i, j \leq t$ ; *with*  $2 \leq t \leq mn$  *where*  $W_i$ 's *are vector subspaces of*  $V$  *over*  $F$  *of infinite dimension over*  $F$ .

Proof is direct and hence left as an exercise to the reader.

Now we proceed onto define the notion of linear algebra using the special interval  $[0, n)$  over a field  $F \subseteq \mathbb{Z}_n$ .

**DEFINITION 2.2 :** *Let*  $V = \{[0, n), +\}$  *be a special interval vector space over a field*  $F \subseteq \mathbb{Z}_n$ .

*If on*  $V$  *we define* ' $\times$ ' *such that*  $(V, \times)$  *is a semigroup, then we define*  $V$  *to be a pseudo special interval linear algebra over*  $F$ .

We will illustrate this situation by some examples.

**Example 2.28:** Let  $V = \{[0, 19), +, \times\}$  is a pseudo special interval linear algebra over the field  $Z_{19}$ .

We see if  $x = 16$  and  $y = 10 \in V$  then  
 $x \times y = 16 \times 10 = 160 = 8 \pmod{19} \in V$ .

If  $x = 0.784$  and  $y = 16 \in V$  then  
 $x \times y = 0.784 \times 16 = 12.544 \in V$ .

Let  $x = 5.02$  and  $y = 18 \in V$

$x \times y = 5.02 \times 18 = 90.36 \pmod{19} = 14.36 \in V$ .

We see  $V$  is also a linear algebra of infinite dimension over  $F$ .

**Example 2.29:** Let  $V = \{[0, 29), +, \times\}$  be the pseudo special interval linear algebra over the field  $F = \{[0, 4, 8] \subseteq Z_{12}$ .

$W = \{\{0, 1, 2, 3, \dots, 11\}, +, \times\} \subseteq V$  is a finite dimensional sublinear algebra of  $V$  over  $F$ .

$V$  has finite dimensional vector subspace which are also linear algebras over  $F$ .

However  $V$  has vector subspace of finite dimension which are not linear algebras over  $F$ .

For take  $P = \{\{0, 0.5, 1, 1.5, 2, 2.5, \dots, 10, 10.5, 11, 11.5\}, +\} \subseteq V$ ,  $P$  is only a vector subspace of  $V$  over  $F$ .

Clearly  $P$  is not a pseudo special sublinear algebra of  $V$  over  $F$  as  $0.5 \times 1.5 = 0.75 \notin P$  for  $0.5$  and  $1.5 \in P$  hence the claim.

Infact we have several such vector subspaces in  $V$  which are not pseudo linear subalgebras of  $V$  over  $F$ .

We call these vector subspaces of this special pseudo interval linear algebra as quasi vector subspaces of  $V$ .

**Example 2.30:** Let  $V = \{[0, 23), +, \times\}$  be a special pseudo interval linear algebra over the field  $F = \mathbb{Z}_{23}$ . Take  $M = \{\{0, 1, 2, 3, \dots, 22\}, +, \times\} \subseteq V$  is a sublinear algebra of  $V$  over  $F$ .

$P = \{\{0, 0.5, 1, 1.5, 2, 2.5, \dots, 20.5, 21, 21.5, 22, 22.5\}, +, \times\} \subseteq V$  is a special quasi vector subspace of  $V$  over  $F$ ; however  $P$  is not closed under product for  $1.5 \in P$  but  $(1.5)^2 = 2.25 \notin P$  hence  $P$  is a special pseudo semilinear subalgebra of  $V$  only a vector subspace of  $V$ .

Let  $W = \{\{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, 1.6, \dots, 22, 22.2, 22.4, 22.6, 22.8\}, +\} \subseteq V$  be a vector quasi subspace of  $V$  and is not a pseudo linear subalgebra of  $V$ .

Thus  $V$  has several vector quasi subspaces which are not linear subalgebras of  $V$ .

**Example 2.31:** Let  $V = \{[0, 7), +, \times\}$  be a special pseudo interval linear algebra over a field  $F = \mathbb{Z}_7$ .

Let  $M_1 = \{0, 1, 2, \dots, 6\} \subseteq V$ .  $M_1$  is a special pseudo interval linear subalgebra of  $V$  over  $F$ .

$M_2 = \{\{0, 0.5, 1, 1.5, 2, 2.5, \dots, 6, 6.5\}, +\} \subseteq V$  is only a special quasi vector subspace of the linear algebra  $V$  over  $F$ .

Clearly if  $x, y \in M_2$ , in general  $x \times y \notin M_2$ , for take  $x = 1.5$  and  $y = 0.5 \in M_2$ .  $x \times y = 1.5 \times 0.5 = 0.75 \notin M_2$ , so  $M_2$  is not a linear subalgebra of  $V$  over  $F$ .

We have several such special quasi vector subspaces of  $V$  which are not pseudo linear subalgebras of the linear algebra  $V$ .

**Example 2.32:** Let  $V = \{[0, 2), \times, +\}$  be the special pseudo interval linear algebra over the field  $F = \mathbb{Z}_2$ .

$P_1 = \{\{0, 0.1, 0.2, \dots, 0.9, 1, 1.1, 1.2, \dots, 1.9\} \subseteq [0, 2), +\} \subseteq V$  is a special pseudo interval quasi vector subspace of  $V$  over  $Z_2$ . Clearly  $P_1$  is not a linear subalgebras of  $V$ .

**Example 2.33:** Let

$V = \{(a_1, a_2, a_3) \mid a_i \in [0, 17), 1 \leq i \leq 3, +, \times\}$  be the pseudo special interval linear algebra over the field  $F = Z_{17}$ .

We have several sublinear algebras as well as quasi vector subspaces of  $V$  over the field  $F$ .

We see  $T = \{(a_1, a_2, 0) \mid a_1 \in [0, 17), a_2 \in \{0, 0.5, 1, 1.5, 2, 2.5, \dots, 16, 16.5\}, +\}$  is an infinite dimensional quasi vector subspace of  $V$  and is not a special linear subalgebra of  $V$ .

Let  $S = \{(a_1, a_2, a_3) \mid a_i \in \{0, 1, 2, \dots, 16\} = Z_{17} \subseteq [0, 17); 1 \leq i \leq 3, +, \times\} \subseteq V$  is a linear subalgebra of  $V$  over  $F = Z_{17}$ . Clearly  $S$  is finite dimension and basis of  $S$  is  $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ . Thus dimension of  $S$  over  $F$  is 3.

Let  $W = \{(0, a_1, a_2) \mid a_1, a_2 \in [0, 17), +, \times\}$  is a linear subalgebra of  $V$  over  $F$  and dimension of  $W$  over  $F$  and dimension of  $W$  over  $F$  is infinite.

$L = \{(a_1, a_2, a_3) \mid a_1, a_2, a_3 \in \{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, \dots, 16, 16.2, 16.4, 16.6, 16.8\} \subseteq [0, 17), +\}$  is a subvector quasi space of  $V$  over  $F$ .

Clearly  $L$  is not closed under  $\times$ , so  $L$  is only a quasi vector subspace of  $V$  over  $F$  so is not a linear subalgebra of  $V$ .

For if  $x = (0.8, 0.4, 0.6) \in L$  then  $x + x = (0.8, 0.4, 0.6) + (0.8, 0.4, 0.6) = (1.6, 0.8, 1.2) \in L$  but  $x \times x = (0.8, 0.4, 0.6) \times (0.8, 0.4, 0.6) = (0.64, 0.16, 0.36) \notin L$ .

So  $L$  is only a quasi vector subspace of  $V$  over  $F$ .



We see  $V$  has several quasi vector subspaces of  $F$  which are not linear subalgebras of  $V$  over  $F$ .

However  $V$  can be written as a direct sum of sublinear algebras of  $V$  over  $F$ .

Let  $B_1 = \{(a_1, 0, 0) \mid a_1 \in [0, 17), +, \times\} \subseteq V$ ,  
 $B_2 = \{(0, a_2, 0) \mid a_2 \in [0, 17), +, \times\} \subseteq V$  and  
 $B_3 = \{(0, 0, a_3) \mid a_3 \in [0, 17), +, \times\} \subseteq V$   
 are all sublinear algebras of  $V$  over  $F$  of infinite dimension.

Further  $B_i \cap B_j = \{(0, 0, 0)\}$  if  $i \neq j$ ,  $1 \leq i, j \leq 3$  and  $B_1 + B_2 + B_3 = V$ .

Thus  $V$  is the direct sum of sublinear algebras over  $F$ .

Let

$C_1 = \{(a_1, 0, 0) \mid a_1 \in \{0, 0.5, 1, 1.5, 2, 2.5, \dots, 15, 15.5, 16, 16.5\} \subseteq [0, 17), +\} \subseteq V$ ,

$C_2 = \{(0, a_2, 0, 0) \mid a_2 \in \{0, 0.5, 1, 1.5, 2, 2.5, \dots, 15, 15.5, 16, 16.5\} \subseteq [0, 17), +\} \subseteq V$  and

$C_3 = \{(0, 0, a_3) \mid a_3 \in \{0, 0.5, 1, 1.5, 2, 2.5, \dots, 15, 15.5, 16, 16.5\} \subseteq [0, 17), +\} \subseteq V$

be the quasi vector subspaces of  $V$  over  $F = \mathbb{Z}_{17}$  as  $C_i$  is not closed under product so  $C_i$ 's are not sublinear algebras of  $V$  over  $F$ .

We have  $C_i \cap C_j = \{(0, 0, 0)\}$  if  $i \neq j$ ,  $1 \leq i, j \leq 3$  yet  $C_1 + C_2 + C_3 = C \subseteq V$  is only a quasi subvector space of  $V$  over  $F$  and is not a direct sum of  $V$ . Infact  $C$  is also finite dimensional over  $F$ .

The special pseudo interval linear algebra has also sublinear algebras such that the sum is not distinct.

For let

$$\begin{aligned} T_1 &= \{(a_1, a_2, 0) \mid a_1, a_2 \in [0, 17), +, \times\} \subseteq V, \\ T_2 &= \{(0, a_1, a_2) \mid a_1, a_2 \in [0, 17), +, \times\} \subseteq V \text{ and} \\ T_3 &= \{(a_1, 0, a_2) \mid a_1, a_2 \in [0, 17), +, \times\} \subseteq V \end{aligned}$$

be special interval linear subalgebras of  $V$  over  $F$ . We see  $T_i \cap T_j \neq \{(0, 0, 0)\}$  even if  $i \neq j$ ,  $1 \leq i, j \leq 3$ .

Further  $V \subseteq T_1 + T_2 + T_3$  so  $V$  is not a direct sum.

**Example 2.34:** Let

$V = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in [0, 38), 1 \leq i \leq 5, +, \times\}$  be the special pseudo interval linear algebra over the field  $F = \{0, 19\} \subseteq \mathbb{Z}_{38}$ .

Clearly  $V$  is an infinite dimensional linear algebra over  $F$ .

Let  $W = \{(a_1, 0, a_2, 0, 0) \mid a_1, a_2 \in [0, 38), +, \times\} \subseteq V$ ;  $W$  is a sublinear algebra of  $V$  over  $F$ .

$T = \{(0, a_1, 0, 0, 0) \mid a_1 \in \{0, 19\}, +, \times\}$  is a sublinear algebra of  $V$  of finite dimension over  $F$ .

$W \cap T = \{(0, 0, 0, 0, 0)\}$ . We see for every  $w \in W$  and for every  $t \in T$  we have  $t \times w = (0, 0, 0, 0, 0)$ .

However  $W + T \subseteq V$  and is again an infinite dimensional sublinear algebra of  $V$  over  $F$ .

Let  $N = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in \{0, 0.5, 1, 1.5, 2, \dots, 18, 18.5, \dots, 37, 37.5\} \subseteq \mathbb{Z}_{19}, 1 \leq i \leq 5, +\} \subseteq V$  is only a finite dimensional quasi vector subspace of  $V$  over  $F$  and is not a linear subalgebra of  $V$  over  $F$ .

**Example 2.35:** Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} \mid a_i \in [0, 6), 1 \leq i \leq 7, +, \times_n \right\}$$

be a special pseudo interval linear algebra of  $V$  over the field  $F = \{0, 3\} \subseteq \mathbb{Z}_6$ . (Clearly  $F \cong \mathbb{Z}_2$ ).  $V$  is infinite dimensional over  $F$ .

Take

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_7 \end{bmatrix} \mid a_i \in \mathbb{Z}_6, 1 \leq i \leq 7, +, \times \right\}$$

is a pseudo linear subalgebra of  $V$  over  $F = \{0, 3\}$ .

Clearly  $M$  is finite dimensional.

$$\text{Let } x = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 4 \\ 5 \\ 1 \\ 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 5 \\ 2 \\ 1 \\ 0 \\ 4 \\ 3 \\ 2 \end{bmatrix} \in M.$$

$$x \times_n y = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 4 \\ 5 \\ 1 \\ 0 \end{bmatrix} \times_n \begin{bmatrix} 5 \\ 2 \\ 1 \\ 0 \\ 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 3 \\ 0 \\ 2 \\ 3 \\ 0 \end{bmatrix} \in M.$$

$$x + y = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 4 \\ 5 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \\ 1 \\ 0 \\ 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 4 \\ 4 \\ 3 \\ 4 \\ 2 \end{bmatrix} \in M.$$

$M$  is finite dimensional over  $F$ .

$$\text{The basis } B \text{ of } M \text{ is } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The dimension of  $M$  over  $F$  is 7.

Let

$$N = \left\{ \begin{bmatrix} a_1 \\ 0 \\ a_2 \\ 0 \\ a_3 \\ 0 \\ a_4 \end{bmatrix} \mid a_i \in \mathbb{Z}_6, 1 \leq i \leq 4, +, \times_n \} \subseteq V$$

be a special pseudo linear subalgebra of  $V$  over  $F$ . Clearly  $N$  is finite dimensional sublinear algebra of  $V$ .

$$\text{The basis of } N \text{ is } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Thus dimension of  $N$  over  $F$  is 4.

Let

$$B = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_7 \end{bmatrix} \mid a_i \in \{0, 2, 4\} \subseteq \mathbb{Z}_6, 1 \leq i \leq 7, +, \times_n \} \subseteq V$$

be a special pseudo interval linear subalgebra of  $V$  over  $F$ .

$B$  is a finite dimensional linear algebra over  $F$ .

$$\text{A basis for } B \text{ is } \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \right\}.$$

Thus dimension of  $B$  over  $F$  is 7. Further  $|B| < \infty$ .

We can write  $V$  as a direct sum of sublinear algebras.

If  $V = W_1 + \dots + W_t$ ,  $t$  can maximum be seven and minimum value for  $t$  is 2.

Let

$$W_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \mid a_1, a_2 \in [0, 6), +, \times_n \} \subseteq V$$

be a pseudo linear subalgebra of  $V$  over  $F$ .

$$W_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_1, a_2 \in [0, 6), +, \times_n \} \subseteq V$$

be a linear pseudo subalgebra of  $V$  over  $F$ .

$$W_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ 0 \\ 0 \end{bmatrix} \mid a_1 \in [0, 6), +, \times_n \} \subseteq V$$

be a linear pseudo subalgebra of  $V$  over  $F$  and

$$W_4 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \end{bmatrix} \mid a_1, a_2 \in [0, 6), +, \times_n \} \subseteq V$$

be a linear pseudo subalgebra of  $V$  over  $F$ .

Clearly

$$W_i \cap W_j = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ if } i \neq j, 1 \leq i, j \leq 4.$$

Further  $V = W_1 + W_2 + W_3 + W_4$ ; thus  $V$  is the direct sum of sublinear pseudo algebras over  $F$ .

Let

$$L_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_1, a_2, a_3 \in [0, 6), +, \times_n \} \subseteq V,$$

$$L_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_2 \\ a_1 \\ 0 \\ 0 \end{bmatrix} \mid a_1, a_2 \in [0, 6), +, \times_n \} \subseteq V \text{ and}$$



$$L_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \end{bmatrix} \mid a_1, a_2 \in [0, 6), +, \times_n \} \subseteq V$$

be three linear pseudo subalgebras of  $V$  over  $F$ .

Clearly

$$L_i \cap L_j = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ if } i \neq j, 1 \leq j, i \leq 3 \text{ and}$$

$V = L_1 + L_2 + L_3$  thus  $V$  is the direct sum of pseudo sublinear algebras of  $V$  over  $F$ .

Let

$$B_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \mid a_1 \in [0, 6), a_2 \in Z_6, +, \times_n \} \subseteq V$$

be a pseudo sublinear algebra of  $V$  over  $F$ . Clearly it is impossible to find more pseudo sublinear algebras so that  $B_1$  can be in the direct sum of  $V$ .

Infact  $B_1$  is infinite dimensional over  $F$ ; we cannot find  $B_i$ 's to make them into a direct sum of  $V$  over  $F$ .

**Example 2.36:** Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \mid a_i \in [0, 11), 1 \leq i \leq 5, +, \times_n \right\}$$

be a special pseudo interval linear algebra over the field  $Z_{11} = F$ .

Clearly  $V$  is of infinite dimension.  $V$  can be written as a direct sum of sublinear algebras.

$V$  has quasi vector subspaces of finite dimension as well as infinite dimension.

For take

$$T_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_1, a_2 \in \{0, 0.5, 1, 1.5, 2, 2.5, \dots, 9, 9.5, 10, 10.5\} \right\}$$

$$\subseteq [0, 11), + \subseteq V$$

is a quasi subvector space of  $V$  over  $F$  and dimension of  $T_1$  over  $F$  is finite.

But

$$T_2 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_1 \in [0, 11) \ a_2 \in \{0, 0.5, 1, 1.5, 2, 2.5, \dots, 9, \right.$$

$$9.5, 10, 10.5\} \subseteq [0, 11), +\} \subseteq V$$

be the quasi vector subspace of  $V$  as if

$$x = \begin{bmatrix} 0.31 \\ 0.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 10 \\ 1.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in T_2$$

$$x \times_n y = \begin{bmatrix} 0.31 \\ 0.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} \times_n \begin{bmatrix} 10 \\ 1.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.1 \\ 0.75 \\ 0 \\ 0 \\ 0 \end{bmatrix} \notin T_2 \text{ as } 0.75 \in \{0, 0.5, 1,$$

$$1.5, 2, 2.5, \dots, 9, 9.5, 10, 10.5, 11, 11.5\} \subseteq [0, 11).$$

Thus  $T_2$  is only a quasi vector subspace and is not a pseudo linear subalgebra over  $F$ .

**Example 2.37:** Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \mid a_i \in [0, 12), 1 \leq i \leq 18, +, \times_n \right\}$$

be the special pseudo interval linear algebra over the field  $F = \{0, 4\} \subseteq Z_{12}$ .

$V$  has quasi vector subspaces which are finite dimensional as well as quasi vector subspaces of infinite dimension.

Take

$$M_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \mid a_1, a_2, a_3 \in [0, 12), 1 \leq i \leq 18, +, \times_n \right\} \subseteq V$$

is a sublinear algebra of  $V$  over  $F$  of infinite dimension over  $F$ .

$$N_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \mid a_1, a_2, a_3 \in Z_{12}, +, \times_n \right\} \subseteq V$$

is a sublinear algebra of  $V$  over  $F$  of finite dimension.

$$N_2 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \mid a_1, a_2, a_3 \in \{0, 0.5, 1, 1.5, 2, 2.5, \dots, \right.$$

$$9, 9.5, 10, 10.5, 11, 11.5\} \subseteq [0, 12), +\} \subseteq V$$

is only a quasi vector subspace of  $V$  as, in  $P_1$  we cannot define product for if

$$x = \begin{bmatrix} 0.5 & 0.5 & 1.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 2.5 & 0.5 & 1.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \in P, \text{ then}$$

$$x \times_n y = \begin{bmatrix} 0.5 & 0.5 & 1.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \times_n \begin{bmatrix} 2.5 & 0.5 & 1.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1.25 & 0.25 & 2.25 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \notin P_1.$$

Thus  $P_1$  is only a quasi vector subspace of  $V$  over  $F$ .

Let

$$R_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \mid a_1 \in [0, 12), \quad a_2, a_3 \in \{0, 0.5, 1, 1.5, \right.$$

$$2, 2.5, \dots, 9, 9.5, 10, 10.5, 11, 11.5\} \subseteq [0, 12), + \} \subseteq V$$

be a special quasi vector subspace of  $V$  over  $F$ .

For if

$$x = \begin{bmatrix} 3.2 & 0.5 & 0.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 5 & 1.5 & 2.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \in R_1 \text{ then}$$

$$x \times_n y = \begin{bmatrix} 3.2 & 0.5 & 0.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \times_n \begin{bmatrix} 5 & 1.5 & 2.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0.75 & 1.25 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \notin R_1.$$

We have infinite dimensional quasi subvector spaces as well as finite dimensional quasi subvector spaces.

We can write  $V$  as a direct sum of sublinear algebras.

This way of representation of  $V$  as a direct sum is not unique.

For

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \mid a_i \in [0, 12), 1 \leq i \leq 6, +, \times_n \right\}$$

$$= W_1 \oplus W_2$$

$$= \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ \vdots & \vdots & \vdots \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \mid a_i \in [0, 12), 1 \leq i \leq 12, +, \times_n \right\}$$

is the direct sum as both  $W_1$  and  $W_2$  are special pseudo sublinear algebras of  $V$  over  $F$  and

$$W_1 \cap W_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}.$$

Similar we can write  $V$  as a direct sum of  $W_1 + \dots + W_t$  pseudo sublinear algebras where  $2 \leq t \leq 18$ .

T can take the maximum value of 18 and minimum of 2.

Let

$$A_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \mid a_1 \in [0, 12), +, \times_n \subseteq V, \right.$$

$$A_2 = \left\{ \begin{bmatrix} 0 & a_2 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \mid a_2 \in [0, 12), +, \times_n \subseteq V, \right.$$

$$A_3 = \left\{ \begin{bmatrix} 0 & 0 & a_3 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \mid a_3 \in [0, 12), +, \times_n \subseteq V \text{ and} \right.$$

so on and

$$A_{18} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & a_{18} \end{bmatrix} \mid a_{18} \in [0, 12), +, \times_n \subseteq V \right.$$

are all special pseudo interval linear subalgebras of  $V$  over the field  $F$ .



Clearly

$$A_i \cap A_j = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \right\}, 1 \leq i, j \leq 18,$$

further  $V = A_1 + A_2 + \dots + A_{18}$ .

Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \right\} \quad a_i \in \{0, 1, 2, \dots, 10, 11\} = Z_{12},$$

$$1 \leq i \leq 18, +, \times_n\}$$

be a special pseudo interval linear subalgebra of  $V$  over  $F$ .

Clearly  $P$  is finite dimensional over  $F$  as a linear pseudo subalgebra of  $V$  over  $F$ .

$$P = P_1 + P_2 + \dots + P_{18}.$$

$$\text{where } P_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \right\} \quad a_1 \in Z_{12}, +, \times_n\} \subseteq V,$$

$$P_2 = \left\{ \begin{bmatrix} 0 & a_2 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \mid a_2 \in Z_{12}, +, \times_n \} \subseteq P \subseteq V, \dots,$$

$$P_{15} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & a_{15} \\ 0 & 0 & 0 \end{bmatrix} \mid a_{15} \in Z_{12}, +, \times_n \} \subseteq P \subseteq V,$$

$$P_{16} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ a_{16} & 0 & 0 \end{bmatrix} \mid a_{16} \in Z_{12}, +, \times_n \} \subseteq P \subseteq V,$$

$$P_{17} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & a_{17} & 0 \end{bmatrix} \mid a_{17} \in Z_{12}, +, \times_n \} \subseteq P \subseteq V,$$

$$P_{18} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 0 & a_{18} \end{bmatrix} \mid a_{18} \in Z_{12}, +, \times_n \} \subseteq P \subseteq V$$

are all sublinear subalgebras of the linear pseudo subalgebra  $P$  over  $F$ .

We see  $P = P_1 + P_2 + \dots + P_{18} \subseteq V$  and

$$P_i \cap P_j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}, i \neq j, 1 \leq i, j \leq 18.$$

$P$  is a subdirect subsum of the sublinear pseudo subalgebras of  $P$  (or  $V$ ) but the direct sum does not give  $V$ .

We can write  $P = P_1 + \dots + P_t$  where  $2 \leq t \leq 18$  but it is subdirect subsum of sublinear algebras. For  $P$  properly contains sublinear algebras of the special interval linear algebra  $V$  over  $F$ .

We see

$$W_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \mid a_1 \in \{0, 0.5, 1, 1.5, 2, \dots, 11, 11.5\} \right\}$$

$$\subseteq [0, 12), +, \times_n \subseteq V$$

is only a special interval quasi vector subspace of  $V$  and is not a sublinear algebra of  $V$  as; if

$$A = \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \in W, \text{ we see}$$

$$A \times_n B = \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \times_n \begin{bmatrix} 2.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3.75 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \notin W_1,$$

thus  $W_1$  is only a quasi vector subspace of  $V$ .

Infact  $V$  has several such quasi vector subspaces some of them are of infinite dimensional and others are finite dimensional.

For take

$$W_2 = \left\{ \begin{bmatrix} a & b & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \mid a \in \{0, 0.5, 1, 1.5, 2, \dots, 11, 11.5\} \text{ and} \right.$$

$$b \in [0, 12), +\} \subseteq V,$$

$W_2$  is only a special quasi vector subspace of  $V$  and it is infinite dimensional over  $F$ .

We see  $W_2$  is not a linear subalgebra of  $V$  over  $F$  as product cannot be defined on  $W_2$ .

$$\text{Let } A = \begin{bmatrix} 0.5 & 3.12 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1.5 & 0.12 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \in W_2.$$

$$A \times_n B = \begin{bmatrix} 0.5 & 3.12 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \times_n \begin{bmatrix} 1.5 & 0.12 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.75 & 0.3744 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \notin W_2 \text{ as } 0.75 \notin \{0, 0.5, 1, 1.5,$$

$$2, \dots, 11, 11.5\} \subseteq [0, 12).$$

Thus  $W_2$  is only a special quasi vector subspace of  $V$  and is not a sublinear algebra.

Further  $W_2$  is infinite dimensional over  $F$ .

$V$  has several such special quasi vector subspace which is infinite dimensional and is not a sublinear algebra of  $V$  over  $F$ .

Even if we do not use the term pseudo it implies the special linear algebras are pseudo special linear algebras from the very context.

Let

$$W_3 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ a_4 & a_5 & a_6 \end{bmatrix} \mid a_1, a_4, a_6 \in [0, 12), a_2, a_3, a_5 \in \{0, 0.2, \right.$$

$$0.4, 0.6, 0.8, 1, 1.2, \dots, 11, 11.2, \dots, 11.8\} \subseteq [0, 12) \} \subseteq V;$$

$W_3$  is a quasi vector subspace of  $V$  over  $F$  and is not a sublinear algebra of  $V$  over  $F$ . Clearly  $W_3$  is also infinite dimensional over  $F$ .

Let

$$M_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \mid a_i \in \mathbb{Z}_{12}; 1 \leq i \leq 18, \times_n, + \} \subseteq V$$

is a special interval linear subalgebra of  $V$  over the field  $F$ .

Clearly  $M$  is finite dimensional over  $F$ .

Let

$$L_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \mid a_i \in \{0, 0.5, 1, 1.5, 2, \dots, 11, \right.$$

$$11.5\} \subseteq [0, 12), +; 1 \leq i \leq 18 \} \subseteq V$$

be a special interval quasi vector subspace of  $V$  over  $F$  and is not a linear subalgebra of  $V$ . Further  $L_1$  is finite dimensional over  $F$ .

**Example 2.38:** Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{30} \\ a_{31} & a_{32} & \dots & a_{40} \end{bmatrix} \mid a_i \in [0, 41), 1 \leq i \leq 40, +, \times_n \} \subseteq V$$

be the special interval linear algebra over the field  $F = Z_{41}$ . Clearly  $V$  is infinite dimensional over  $F$ .

Let

$$M_1 = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{30} \\ a_{31} & a_{32} & \dots & a_{40} \end{bmatrix} \mid a_i \in Z_{41}, 1 \leq i \leq 40, +, \times_n \} \subseteq V$$

be a special pseudo interval linear subalgebra of  $V$  over  $F$  of finite dimension.

This  $M_1$  has several special pseudo interval linear subalgebras of finite dimension over  $F = Z_{41}$ .

Let

$$N_1 = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{30} \\ a_{31} & a_{32} & \dots & a_{40} \end{bmatrix} \mid a_i \in \{0, 0.5, 1, 1.5, 2, \dots, 11, 11.5\} \right. \\ \left. \subseteq [0, 41), 1 \leq i \leq 40, + \} \subseteq V$$

be a special quasi interval subvector space of  $V$  over  $F$ . Clearly  $N_i$  is not a linear subalgebra.

For if

$$A = \begin{bmatrix} 0.5 & 0.5 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 1.5 & 0 & 0 & \dots & 0 \\ 0 & 1.5 & 0 & \dots & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1.5 & 0.5 & 0 & \dots & 0 \\ 2 & 0 & 0 & \dots & 0 \\ 2.5 & 0 & 0 & \dots & 7.5 \\ 0 & 0.5 & 0 & \dots & 0 \end{bmatrix} \in N_1.$$

Then

$$\begin{aligned} A \times_n B &= \begin{bmatrix} 0.5 & 0.5 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 1.5 & 0 & 0 & \dots & 0 \\ 0 & 1.5 & 0 & \dots & 0 \end{bmatrix} \times_n \begin{bmatrix} 1.5 & 0.5 & 0 & \dots & 0 \\ 2 & 0 & 0 & \dots & 0 \\ 2.5 & 0 & 0 & \dots & 7.5 \\ 0 & 0.5 & 0 & \dots & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0.75 & 0.25 & 0 & \dots & 0 \\ 2 & 0 & 0 & \dots & 0 \\ 3.75 & 0 & 0 & \dots & 0 \\ 0 & 0.75 & 0 & \dots & 0 \end{bmatrix} \notin N_1. \end{aligned}$$

Thus  $N_1$  is not special pseudo linear subalgebra of  $V$  over  $F$   $= Z_{41}$  only a special quasi vector subspace of  $V$  over  $F$  and is of finite dimension over  $F$ .

$$L_1 = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{bmatrix} \mid a_1, a_2 \in \{0, 0.5, 1, 1.5, 2, \dots, 11, 11.5, \dots, 39, 39.5, 40, 40.5\} \subseteq Z_{41} \right\} \subseteq V;$$



$L_1$  is not a sublinear algebra over  $F = \mathbb{Z}_{41}$  only a quasi vector subspace of  $V$  over  $\mathbb{Z}_{41}$  and is finite dimensional over  $\mathbb{Z}_{41}$ .

We have several such quasi vector subspaces some of them finite dimensional and some are infinite dimension.

Now all the special interval linear algebras given by us are of infinite dimensional and infact were commutative.

Now we proceed onto give examples non commutative special interval linear algebras.

**Example 2.39:** Let

$$V = \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \mid a_1, a_2, a_3, a_4 \in [0, 13), +, \times \right\}$$

be the special pseudo interval linear algebra over the field  $F = \mathbb{Z}_{13}$ . Clearly if  $A, B \in V$  we see  $A \times B \neq B \times A$  in general.

Thus  $V$  is a non commutative linear algebra of infinite dimension over  $F = \mathbb{Z}_{13}$ .

Let

$$P_1 = \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \mid a_i \in \{0, 0.5, 1, 1.5, \dots, 12, 12.5, +\} \subseteq V \right\}$$

is a special quasi vector subspace of  $V$  over  $F$ .

Infact we have for some  $A, B$  in  $P_1$ ;  $A \times B \in V$  but is not in  $P_1$  in general.

$$x = \begin{bmatrix} 1.5 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix} \in P_1.$$

$$x \times y = \begin{bmatrix} 1.5 & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.75 & 0 \\ 0 & 0 \end{bmatrix} \notin P_1.$$

Thus  $P_1$  is not a linear subalgebra only a quasi vector subspace of  $V$ .

$$\text{Let } A = \begin{bmatrix} 0.2 & 0 \\ 1 & 0.5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0.5 & 6 \\ 0.2 & 1 \end{bmatrix} \in V.$$

$$\begin{aligned} \text{We find } A \times B &= \begin{bmatrix} 0.2 & 0 \\ 1 & 0.5 \end{bmatrix} \times \begin{bmatrix} 0.5 & 6 \\ 0.2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0.10 & 1.2 \\ 0.5 + .10 & 6 + 0.5 \end{bmatrix} \\ &= \begin{bmatrix} 0.10 & 1.2 \\ 0.15 & 6.5 \end{bmatrix} \quad \dots \text{ (I)} \end{aligned}$$

$$\begin{aligned} B \times A &= \begin{bmatrix} 0.5 & 6 \\ 0.2 & 1 \end{bmatrix} \times \begin{bmatrix} 0.2 & 0 \\ 1 & 0.5 \end{bmatrix} \\ &= \begin{bmatrix} 0.1 + 6 & 3 \\ 0.04 + 1 & 0.5 \end{bmatrix} = \begin{bmatrix} 6.1 & 3 \\ 1.04 & 0.5 \end{bmatrix} \quad \dots \text{ (II)} \end{aligned}$$

Clearly I and II are distinct hence  $V$  is only a non commutative linear algebra over  $F$ .

**Example 2.40:** Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \mid a_i \in [0, 22), 1 \leq i \leq 16, +, \times \right\}$$

be the special interval linear algebra over the field  $F = \{0, 11\}$ .

Clearly  $V$  is a non commutative linear algebra. This linear algebra is infinite dimensional.

This has such pseudo special linear algebras which are commutative both finite or infinite dimension.

$$W_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid a_1 \in [0, 22), +, \times \right\}$$

is a sublinear algebra of  $V$  which is commutative and is infinite dimensional.

For if

$$M_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid a_1 \in \mathbb{Z}_{22}, +, \times \right\} \subseteq V$$

be a linear subalgebra of  $V$  over  $F$ .

Clearly  $M_1$  is commutative over  $F$  and is finite dimensional over  $F$ .

$$N_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{bmatrix} \mid a_i \in [0, 22), 1 \leq i \leq 4, +, \times \right\} \subseteq V$$

be a pseudo linear subalgebra over the field  $F = \{0, 11\}$ .

$N_1$  is a commutative pseudo linear subalgebra of infinite dimension over  $F$ .

$$M_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{bmatrix} \mid a_i \in \mathbb{Z}_{22}, 1 \leq i \leq 4, +, \times \} \subseteq V$$

is a linear subalgebra which is commutative and is of finite dimension over  $F$ .

$$N_2 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & 0 & a_6 & 0 \\ 0 & 0 & 0 & 0 \\ a_7 & a_8 & a_9 & a_{10} \end{bmatrix} \mid a_i \in [0, 22), 1 \leq i \leq 10, +, \times \} \subseteq V$$

is a sublinear algebra which is non commutative and is of infinite dimension over  $F$ .

Let

$$T_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid a_1 \in \{0, 0.5, 1, 1.5, 2, 2.5, \dots, 21, 21.5\} \right.$$

$$\subseteq [0, 22), +, \times \} \subseteq V$$

be a quasi subvector space as if  $A, B \in T$ ,  $A \times B \notin T_1$  is general.

Hence  $T_1$  is commutative and finite dimensional quasi subvector space of  $V$ .

Let

$$P_1 = \left\{ \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ a_3 & 0 & 0 & a_7 \\ 0 & 0 & 0 & 0 \\ 0 & a_4 & a_5 & a_6 \end{bmatrix} \mid a_i \in \{0, 0.5, 1, 1.5, 2, 2.5, \dots, \right.$$

$$21, 21.5\} \subseteq [0, 22), 1 \leq i \leq 7, +, \times\} \subseteq V$$

be a quasi vector subspace of  $V$  and is not a sublinear algebra for if  $A, B \in P_1$  we see  $A \times B \notin P_1$ . Thus  $P_1$  is a quasi vector subspace of finite dimension over  $F$ .

**Example 2.41:** Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \mid a_i \in [0, 14), 1 \leq i \leq 9, +, \times\} \right.$$

be a non commutative special interval linear algebra over  $F = \{0, 7\}$ .  $V$  is infinite dimensional.

As linear algebras  $V$  has sublinear algebras.  $V$  has quasi subvector subspace over  $F$ .

Let

$$M_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \mid a_i \in [0, 14), 1 \leq i \leq 3, +, \times\} \subseteq V$$

be an infinite dimensional special pseudo interval sublinear algebra over  $F$ .

Clearly  $M_1$  is a commutative pseudo linear algebra over  $F$ .

$$M_2 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \mid a_i \in \{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, \dots, 13, \right.$$

$$13.2, 13.4, 13.6, 13.8\} \subseteq [0, 14), 1 \leq i \leq 3, +, \times \} \subseteq V$$

is only a quasi vector subspace of  $V$  but it is not a linear subalgebra of  $V$  as if  $A, B \in M_2$  we see  $A \times B \notin M_2$ .

However  $V$  is a finite dimensional quasi vector subspace of  $V$  which is commutative over  $F$ .

$$\text{Let } A = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.8 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0.8 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 2.4 \end{bmatrix} \in M_2;$$

$$\begin{aligned} A \times B &= \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.8 \end{bmatrix} \times \begin{bmatrix} 0.8 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 2.4 \end{bmatrix} \\ &= \begin{bmatrix} 0.16 & 0 & 0 \\ 0 & .72 & 0 \\ 0 & 0 & 1.92 \end{bmatrix} \notin M_2. \end{aligned}$$

Hence the claim.

Let

$$A = \left\{ \begin{bmatrix} a_1 & a_2 & 0 \\ 0 & a_3 & 0 \\ a_4 & 0 & a_5 \end{bmatrix} \mid a_4, a_i \in [0, 14) \ a_2, a_3, a_5 \in \{0, 0.5, 1, 1.5, \right.$$

$$2, \dots, 12, 12.5, 13, 13.5\} \subseteq [0, 14), +, \times \} \subseteq V$$

be the special quasi vector subspace of  $V$  and  $A$  is not a linear subalgebra.

Certainly  $A$  is only a quasi vector subspace of  $V$  of infinite dimension over  $F$ .

Thus  $V$  has both finite and infinite dimensional quasi vector subspaces.

$V$  has both finite and infinite dimensional linear subalgebras over  $F$ .

The question of non commutativity does not arise in case of quasi vector subspaces; however in case of linear subalgebras we may have them to be commutative or otherwise.

**Example 2.42:** Let

$V = \{(a_1 | a_2 a_3 a_4 | a_5) | a_i \in [0, 5), 1 \leq i \leq 5, +, \times\}$  be a special interval linear algebra over the field  $F = \mathbb{Z}_5$ .

$V$  has sublinear algebras as well as quasi vector spaces of infinite and finite dimension.

$M_1 = \{(a_1 | 0 0 0 | 0) | a_1 \in [0, 5), +, \times\} \subseteq V$  is a sublinear algebra of infinite dimension over  $F$ .

$N_1 = \{(a_1 | 0 0 0 | 0) | a_1 \in \mathbb{Z}_5, +, \times\} \subseteq V$  is a sublinear algebra of finite dimension over  $F$ .

$W_1 = \{(a_1 | 0 0 0 | 0) | a_1 \in \{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, \dots, 4.2, 4.4, 4.6, 4.8\} \subseteq [0, 5), +, \times\} \subseteq V$  is only a quasi vector space we see if

$X = \{(0.2 | 0 0 0 | 0)\}$  and  $Y = (0.8 | 0 0 0 | 0) \in W_1$  then  $X \times Y = (0.2 | 0 0 0 | 0) \times (0.8 | 0 0 0 | 0) = (0.16 | 0 0 0 | 0) \notin W_1$ , so  $W_1$  is only a special quasi vector subspace of finite dimension over  $F$  and is not a sublinear algebra over  $F$ .

Let

$W_2 = \{(a_1 \mid a_2 \ 0 \ 0 \mid a_3) \mid a_1 \ a_2 \in \mathbb{Z}_5 \text{ and } a_3 \in [0, 5), +, \times\} \subseteq V$  be the special interval linear subalgebra of  $V$  over  $F$ .

Clearly  $W_2$  is of infinite dimensional over  $F$ .

Let  $L_1 = \{(0 \mid a_1 \ a_2 \ a_3 \mid 0) \mid a_i \in \mathbb{Z}_5, 1 \leq i \leq 3\} \subseteq V$  be linear subalgebra of  $V$  over  $F$ . Clearly  $L_1$  is of finite dimension over  $F$ .

Let  $S_1 = \{(0 \mid a_1 \ a_2 \ a_3 \mid 0) \mid a_i \in \{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, \dots, 4.2, 4.4, 4.6, 4.8\} \subseteq [0, 5), 1 \leq i \leq 3\} \subseteq V$  be a quasi subvector space of  $V$  over  $F$  and is not a linear algebra for if  $X = (0 \mid 0.4, 0.8, 1.2 \mid 0)$  and  $Y = (0 \mid 0.2, 0.8, 1.2 \mid 0) \in S_1$ , then  $X \times Y = (0 \mid 0.4, 0.8, 1.2 \mid 0) \times (0 \mid 0.2, 0.8, 1.2 \mid 0) = (0 \mid 0.08, 0.64, 1.44 \mid 0) \notin S_1$ , so  $S_1$  is only a quasi vector subspace of  $V$  and is not a linear subalgebra of  $V$  over  $F$ .

**Example 2.43:** Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \mid a_i \in [0, 12), 1 \leq i \leq 6, +, \times_n \right\}$$

be the special interval linear algebra over  $F = \{0, 4, 8\} \subseteq \mathbb{Z}_{12}$ . Clearly  $V$  is commutative and is infinite dimensional over  $F$ .

$V$  has sublinear algebras of finite as well as infinite dimension.  $V$  also has quasi vector subspaces of both finite and infinite dimension over  $F$ .



Let

$$M_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \frac{a_3}{0} \\ 0 \\ \overline{0} \end{bmatrix} \mid a_i \in [0, 12), 1 \leq i \leq 3, +, \times_n \} \subseteq V$$

is a pseudo special sublinear algebra of  $V$  over  $F$ .

Clearly dimension of  $M_1$  is of infinite cardinality over  $F$ .

$$N_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \frac{a_3}{0} \\ 0 \\ \overline{0} \end{bmatrix} \mid a_i \in Z_{12}, 1 \leq i \leq 3, +, \times_n \} \subseteq V$$

is a pseudo special sublinear algebra of  $V$  over  $F$ . Clearly dimension of  $N_1$  over  $F$  is finite.

$$L_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \frac{a_3}{0} \\ 0 \\ \overline{0} \end{bmatrix} \mid a_i \in \{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, \dots,$$

$$11.2, \dots, 11.8\} \subseteq [0, Z_{12}), 1 \leq i \leq 3, +, \} \subseteq V$$

is only a quasi vector subspace of  $V$  over  $F$ .

For if  $A, B \in L_1$  in general  $A \times_n B \notin V$ .

$$\text{Take } A = \begin{bmatrix} 0.4 \\ 0.8 \\ \frac{0.6}{0} \\ 0 \\ 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1.2 \\ 0.6 \\ \frac{0.4}{0} \\ 0 \\ 0 \end{bmatrix} \text{ in } L_1.$$

$$A \times_n B = \begin{bmatrix} 0.4 \\ 0.8 \\ \frac{0.6}{0} \\ 0 \\ 0 \end{bmatrix} \times_n \begin{bmatrix} 1.2 \\ 0.6 \\ \frac{0.4}{0} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.48 \\ 0.48 \\ \frac{0.24}{0} \\ 0 \\ 0 \end{bmatrix} \notin L_1 \text{ as } 0.48, 0.24 \notin \{0, 0.2,$$

$$0.4, 0.6, 0.8, 1, 1.2, 1.4, 1.6, 1.8, 2, \dots, 11, 11.2, 11.4,$$

$$11.6, 11.8\} \subseteq [0, Z_{12}).$$

$$A = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ \frac{0}{a_2} \\ 0 \\ \frac{0}{a_3} \end{bmatrix} \mid a_1 \in [0, 12), a_2, a_3 \in \{0, 0.5, 1, 1.5, \dots, 11,$$

$$11.5\} \subseteq [0, 12), +, \times_n\} \subseteq V;$$

$A$  is only a quasi vector subspace of  $V$  and is not a pseudo special linear subalgebra of  $V$  over  $F$  for  $\times_n$  is not defined on  $A$ .

$$\text{Take } x = \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 0.5 \\ 0 \\ 1.5 \end{bmatrix} \text{ and } y = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1.5 \\ 0 \\ 0.5 \end{bmatrix} \in A.$$

$$x \times_n y = \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 0.5 \\ 0 \\ 1.5 \end{bmatrix} \times_n \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1.5 \\ 0 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0.75 \\ 0 \\ 0.75 \end{bmatrix} \notin A \text{ as } 0.75 \notin \{0, 0.5,$$

$$1, 1.5, \dots, 11, 11.5\} \subseteq [0, 12).$$

Thus  $A$  is only quasi vector subspace of  $V$  over  $F$ .

$A$  is an infinite dimensional quasi vector subspace of  $V$  over  $F$ .

Hence we have quasi subspaces of finite and infinite dimension and similarly sublinear algebras of infinite and finite dimension.

**Example 2.44:** Let

$$V = \left\{ \begin{bmatrix} \begin{array}{ccc} a_1 & a_2 & a_3 \\ \hline a_4 & a_5 & a_6 \\ \hline a_7 & a_8 & a_9 \\ \hline a_{10} & \dots & a_{12} \\ a_{13} & \dots & \dots \\ a_{16} & \dots & \dots \\ \hline a_{19} & \dots & \dots \\ \hline a_{22} & \dots & \dots \\ a_{25} & \dots & \dots \\ a_{28} & a_{29} & a_{30} \end{array} \end{bmatrix} \mid a_i \in [0, 13), 1 \leq i \leq 30, +, \times_n \right\}$$

be a special pseudo interval linear algebra over the field  $Z_{13}$ .

This linear algebra is commutative and is infinite dimensional over  $Z_{13} = F$ .

$V$  has quasi subspaces of finite as well as infinite dimension over  $F$ . Infact  $V$  has both sublinear algebras of finite and infinite dimension over  $F$ . Take

$$P_1 = \left\{ \begin{bmatrix} \begin{array}{ccc} a_1 & a_2 & a_3 \\ \hline a_4 & a_5 & a_6 \\ \hline a_7 & a_8 & a_9 \\ \hline a_{10} & \dots & a_{12} \\ a_{13} & \dots & \dots \\ a_{16} & \dots & \dots \\ \hline a_{19} & \dots & \dots \\ \hline a_{22} & \dots & \dots \\ a_{25} & \dots & \dots \\ a_{28} & a_{29} & a_{30} \end{array} \end{bmatrix} \mid a_i \in Z_{13}, 1 \leq i \leq 30, +, \times_n \} \subseteq V;$$

$P_1$  is a sublinear algebra of finite dimension over  $F$ .

Let

$$P_2 = \left\{ \left[ \begin{array}{ccc|ccc|ccc} a_1 & a_2 & a_3 & & & & & & \\ \hline 0 & 0 & 0 & & & & & & \\ \hline 0 & 0 & 0 & & & & & & \\ \hline 0 & 0 & 0 & & & & & & \\ \hline 0 & 0 & 0 & & & & & & \\ \hline 0 & 0 & 0 & & & & & & \\ \hline 0 & 0 & 0 & & & & & & \\ \hline 0 & 0 & 0 & & & & & & \\ \hline 0 & 0 & 0 & & & & & & \end{array} \right] \mid a_i \in [0, 13), 1 \leq i \leq 3, +, \times_n \} \subseteq V$$

be a special interval linear subalgebra of  $V$  over  $F$ .

However  $P_2$  is infinite dimensional sublinear algebra over  $F$ .

Let

$$P_3 = \left\{ \left[ \begin{array}{ccc|ccc|ccc} a_1 & a_2 & a_3 & & & & & & \\ \hline 0 & 0 & 0 & & & & & & \\ \hline 0 & 0 & 0 & & & & & & \\ \hline 0 & 0 & 0 & & & & & & \\ \hline 0 & 0 & 0 & & & & & & \\ \hline 0 & 0 & 0 & & & & & & \\ \hline 0 & 0 & 0 & & & & & & \\ \hline 0 & 0 & 0 & & & & & & \\ \hline 0 & 0 & 0 & & & & & & \end{array} \right] \mid a_1, a_2, a_3 \in \{0, 0.5, 1, 1.5, \dots, 12, 12.5\} \subseteq [0, 13), +, \times \} \subseteq V$$

be a special interval quasi vector subspace of  $V$  over  $F = Z_{13}$ . Clearly  $P_3$  is not sublinear algebra over  $F$ . Dimension of  $P_3$  over  $F$  is finite dimensional. Let

$$P_4 = \left\{ \left[ \begin{array}{ccc|ccc|ccc|ccc} a_1 & a_2 & a_3 & & & & & & & & & \\ 0 & 0 & 0 & & & & & & & & & \\ 0 & 0 & 0 & & & & & & & & & \\ \hline 0 & 0 & 0 & & & & & & & & & \\ 0 & 0 & 0 & & & & & & & & & \\ 0 & 0 & 0 & & & & & & & & & \\ \hline 0 & 0 & 0 & & & & & & & & & \\ 0 & 0 & 0 & & & & & & & & & \\ 0 & 0 & 0 & & & & & & & & & \\ 0 & 0 & 0 & & & & & & & & & \end{array} \right] \mid a_1, a_2, a_3 \in \{0, 0.5, 1, 1.5, \dots, 12, 12.5\} \subseteq$$

$$[0, 13), +, \times \} \subseteq V$$

be a special interval quasi vector subspace of  $V$  and is not a sublinear algebra over  $F$ .

$$P_5 = \left\{ \left[ \begin{array}{ccc|ccc|ccc|ccc} a_1 & a_2 & a_3 & & & & & & & & & \\ a_4 & a_5 & a_6 & & & & & & & & & \\ 0 & 0 & 0 & & & & & & & & & \\ \hline 0 & 0 & 0 & & & & & & & & & \\ 0 & 0 & 0 & & & & & & & & & \\ 0 & 0 & 0 & & & & & & & & & \\ \hline 0 & 0 & 0 & & & & & & & & & \\ 0 & 0 & 0 & & & & & & & & & \\ 0 & 0 & 0 & & & & & & & & & \\ 0 & 0 & 0 & & & & & & & & & \end{array} \right] \mid a_i \in Z_{13}, a_j \in [0, 13), 1 \leq i \leq 3, 1 \leq j \leq 3,$$

$$+, \times_n \} \subseteq V$$

is a special interval linear algebra over the field  $F = Z_{13}$  of infinite dimension over  $F$ .

Thus we can have several such special interval linear subalgebras both of finite and infinite dimension over  $F = \mathbb{Z}_{13}$ .

**Example 2.45:** Let

$$V = \left\{ \left( \begin{array}{c|c|c|c|c} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & a_{20} \\ a_{21} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & a_{30} \end{array} \right) \mid a_i \in [0, 41), 1 \leq i \leq 30, +, \times_n \right\}$$

be the special interval linear algebra over the field  $F = \mathbb{Z}_{41}$ .

$V$  has several quasi subvector spaces of finite and infinite dimension over  $F$ .  $V$  has also special interval linear subalgebras over  $F$ . Thus this study leads to several interesting results.

Now we can define linear algebras and vector spaces using polynomials.

We call

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in S = [0, n), \right.$$

$n < \infty$  the special interval  $\}$  to be the polynomial ring over the special interval ring  $([0, n), +, \times)$ . We see if  $p(x), q(x) \in S[x]$  then  $p(x) \times q(x) \in S[x]$ .

These special pseudo interval rings  $\{[0, n), +, \times\}$  has been introduced earlier in the book [11]. These rings are infact pseudo interval integral domains if  $n$  is a prime; only in case of  $n$  a non prime  $S$  will have zero divisors.

As  $n > 1$ , clearly  $1 \in [0, n)$  serves as the identity with respect to multiplication.  $S[x]$  the special interval pseudo polynomial ring can have zero divisor only if  $S$  is built using  $[0, n)$ ,  $n$  a non prime.

We will illustrate this situation by some examples.

**Example 2.46:** Let

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in [0, 23), + \right\}$$

be the special interval vector space over the field  $Z_{23}$ .

Let

$$P[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in Z_{23} \right\} \subseteq S[x]$$

be a special interval vector subspace of  $S[x]$  of infinite dimension. The basis for  $P[x]$  is  $\{1, x, x^2, \dots, x^n, \dots, n \rightarrow \infty\}$ .

However  $S[x]$  is infinite dimensional but has a different set of basis.

Let

$$M[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in \{0, 0.5, 1, 1.5, 2, 2.5, \dots, 22.5\} \subseteq [0, 23) \right\} \subseteq S[x]$$

be an infinite dimensional vector subspace of  $V$  over  $F = Z_{23}$ .

Clearly the dimension of  $V = S[x]$  over  $F$  is different from the dimension of  $M[x] \subseteq V$  over  $F$ .

Let

$$T[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in [0, 23) \right\} \subseteq S[x]$$



be a subspace of  $S[x]$  which is of infinite dimension over  $F$ .

**Example 2.47:** Let

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in [0, 62) \right\}$$

be the special interval polynomial vector space over the field  $F = \{0, 31\}$ .  $S[x]$  has several subspaces both of finite and infinite dimension.

Take

$$P[x] = \left\{ \sum_{i=0}^{12} a_i x^i \mid a_i \in \{0, 31\}, 0 \leq i \leq 12, + \right\} \subseteq S[x]$$

a subspace of  $S[x]$  and is finite dimensional for  $B = \{31, 31x, 31x^2, \dots, 31x^{12}\} \subseteq P[x]$  is a basis of  $P[x]$  over  $F = \{0, 31\}$ .

Clearly dimension of  $P[x]$  is 13 over  $F$ .

We take

$$T[x] = \left\{ \sum_{i=0}^4 a_i x^i \mid a_i \in \{0, 31\}, 0 \leq i \leq 4 \right\} \subseteq S[x]$$

a vector subspace of  $S[x]$  and  $T[x]$  is finite dimensional over  $F$  and dimension of  $T[x]$  is 5 over  $F$  and the basis of  $T[x]$  is given by  $B = \{31, 31x, 31x^2, 31x^3, 31x^4\}$ . Thus we have several such subspaces of finite dimension over  $F$ .

Let

$$D[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in \{0, 2, 4, 6, 8, \dots, 60\} \subseteq \mathbb{Z}_{62} \right\} \subseteq S[x]$$

be a infinite dimensional vector subspace of  $S[x]$  over  $F$ .

Thus  $S[x]$  the special interval polynomial vector space has both finite and infinite dimensional vector subspaces over  $F$ .

If in  $S[x]$  we can define the notion of product then we define  $S[x]$  to be the special interval polynomial pseudo linear algebra over the field  $F$ .

To this end we will give some more examples.

**Example 2.48:** Let

$$S[x] = \left\{ \sum_{i=0}^{21} a_i x^i \mid a_i \in [0, 41), 0 \leq i \leq 21 \right\}$$

be the special interval polynomial vector space over  $F = \mathbb{Z}_{41}$ . Clearly  $S[x]$  is not a special interval polynomial linear pseudo algebra as  $\times$  cannot be defined on  $S[x]$ .

Let  $p(x) = x^{20} + 3x^2 + 1$  and  $q(x) = 3x^{12} + 8x + 31 \in S[x]$

$$\begin{aligned} p(x) \times q(x) &= (x^{20} + 3x^2 + 1) \times (3x^{12} + 8x + 31) \\ &= 3x^{32} + 9x^{14} + 3x^{12} + 8x^{21} + 8x^2 + 24x^3 + 93x^2 + 31x^{20} + 31 \\ &= 3x^{32} + 8x^{21} + 31x^{20} + 9x^{14} + 3x^{12} + 24x^3 + 19x^2 + 31 \notin S[x]. \end{aligned}$$

This product cannot be defined in  $S[x]$ , so  $S[x]$  is only special interval polynomial vector space and not a special interval pseudo linear algebra.

**Example 2.49:** Let

$$S[x] = \left\{ \sum_{i=0}^7 a_i x^i \mid a_i \in [0, 12), 0 \leq i \leq 7, + \right\}$$

be a special interval vector space of polynomials over the field  $F = \{0, 8, 4\} \subseteq \mathbb{Z}_{12}$ .  $S[x]$  is only a vector space and not a special pseudo linear algebra.

$S[x]$  is infinite dimensional over  $F$ . However  $S[x]$  has finite dimensional subspaces as well as infinite dimensional subspaces.

Let

$$P[x] = \left\{ \sum_{i=0}^7 a_i x^i \mid a_i \in \{0, 2, 4, 6, 8, 10\} \subseteq \mathbb{Z}_{12}, 0 \leq i \leq 7 \right\} \subseteq S[x]$$

be a finite dimensional vector subspace of  $S[x]$  over  $F$ .

Let

$$T[x] = \left\{ \sum_{i=0}^7 a_i x^i \mid a_i \in \{0, 1, 0.5, 1.5, 2, 2.5, 3, 3.5, \dots, 11, 11.5\} \right. \\ \left. \subseteq [0, 12), 0 \leq i \leq 7 \right\} \subseteq S[x]$$

be a finite dimensional polynomial vector subspace of  $S[x]$  over  $F = \{0, 8, 4\} \subseteq \mathbb{Z}_{12}$ .

Let

$$V[x] = \left\{ \sum_{i=0}^3 a_i x^i \mid a_i \in [0, 12), 0 \leq i \leq 3 \right\} \subseteq S[x]$$

is an infinite dimensional vector subspaces of  $S[x]$  over  $F$ .

Thus  $S[x]$  has both finite and infinite dimensional polynomial vector subspaces. However  $S[x]$  is never a pseudo special linear algebra.

Inview of all these we just state the following theorem the proof of which is direct.

**THEOREM 2.5:** *Let  $V$  be a special interval pseudo linear algebra defined over a field  $F$ .  $V$  is always a special interval vector space over  $F$  however every special interval vector space defined over a field  $F$  in general is not a special interval pseudo linear algebra.*

**Example 2.50:** Let

$$V = S[x] = \left\{ \sum_{i=0}^{12} a_i x^i \mid a_i \in [0, 7), 0 \leq i \leq 12 \right\}$$

be the special pseudo interval polynomial vector space over the field  $Z_7$ . Clearly  $V$  is never a linear algebra.

Dimension of  $S[x]$  over  $F$  is infinite. However  $S[x]$  has subspaces of finite dimension. For take

$$T[x] = \left\{ \sum_{i=0}^{12} a_i x^i \mid a_i \in Z_7, 0 \leq i \leq 12 \right\} \subseteq V$$

is a subspace of  $V$  but  $T[x]$  is of finite dimension over  $F = Z_7$ .

Let

$$W[x] = \left\{ \sum_{i=0}^5 a_i x^i \mid a_i \in \{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, \dots, 6, 6.2, 6.4, 6.6, 6.8\} \subseteq [0, 7), 0 \leq i \leq 5 \right\} \subseteq V;$$

$W[x]$  is a finite dimensional vector subspace of  $S[x]$  over  $F$ .

$$B[x] = \left\{ \sum_{i=0}^5 a_i x^i \mid a_i \in [0, 7), 0 \leq i \leq 5 \right\} \subseteq S[x]$$

is an infinite dimensional vector subspace of  $x$  over the field  $F$ .

However we can have special interval polynomial pseudo linear algebras over a field.

We will illustrate this situation by a few examples.

**Example 2.51:** Let

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in [0, 23), +, \times \right\}$$

is a special interval polynomial pseudo linear algebra over the field  $F = Z_{23}$ .

$S[x]$  has quasi vector subspaces which are not linear subalgebras both of finite and infinite dimension over  $F = Z_{23}$ .

Also  $S[x]$  has linear pseudo subalgebras all of them are infinite dimensional over  $F$ .

Let

$$\begin{aligned} P[x] &= \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in \{0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, \dots, 22, 22.5\} \right. \\ &\quad \left. \subseteq [0, 23) \right\} \subseteq S[x] \end{aligned}$$

be a subspace of  $S[x]$  over  $F$ . Clearly  $P[x]$  is an infinite dimensional quasi vector subspace of  $S[x]$  over  $F$ .

$P[x]$  is not a pseudo linear subalgebra of  $S[x]$  over  $F$ .

Let

$$T[x] = \left\{ \sum_{i=0}^{20} a_i x^i \mid a_i \in Z_{23}, 0 \leq i \leq 20, + \right\} \subseteq S[x]$$

be a finite dimensional quasi subset vector space over  $F$ .

Clearly  $T[x]$  is not a pseudo sublinear algebra.

Infact

$$B[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in \mathbb{Z}_{23}, +, \times \right\} \subseteq S[x]$$

is a pseudo linear subalgebra of  $S[x]$  over  $F$  and dimension of  $B[x]$  is infinite over  $F$ .

$$D[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in [0, 23), +, \times \right\} \subseteq S[x]$$

is a linear pseudo subalgebra of infinite dimension over  $F$ .

Clearly  $D[x]$  has uncountable infinite basis.

**Example 2.52:** Let

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in [0, 34), +, \times \right\}$$

be a special interval pseudo linear algebra over the field  $F = \{0, 17\} \subseteq \mathbb{Z}_{34}$ .

$S[x]$  has quasi vector subspaces also linear pseudo subalgebras of finite and infinite dimension.

$$\text{Let } T[x] = \left\{ \sum_{i=0}^5 a_i x^i \mid a_i \in \{0, 17\}, 0 \leq i \leq 5, + \right\} \subseteq S[x]$$

be a special quasi vector subspace of  $V$  over  $F$  and dimension of  $T[x]$  is finite. However  $T[x]$  is not closed under product.

$$L[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in \mathbb{Z}_{34}, +, \times \right\} \subseteq V$$

is a special interval pseudo linear subalgebra of  $V$  over  $F$ .

Clearly  $L[x]$  is infinite dimensional over  $F$ .

$$M[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in \{0, 0.5, 1, 1.5, 2, 2.5, \dots, 33.5\}, + \right\}$$

is only a quasi vector subspace over  $F$  and it is not a pseudo linear subalgebra as if  $p(x) = 0.5x^3 + 1.5x^2 + 0.5$  and  $q(x) = 1.5x^2 + 0.5x + 0.5 \in M[x]$ .

$$\begin{aligned} p(x) \times q(x) &= (0.5x^3 + 1.5x^2 + 0.5) \times (1.5x^2 + 0.5x + 0.5) \\ &= 0.75x^5 + 2.25x^4 + 0.75x^2 + 0.25x^4 + 0.75x^2 + 0.5 \\ &= 0.75x^5 + 2.5x^4 + 0x^3 + 1.5x^2 + 0.25x + 0.25 \notin M[x]. \end{aligned}$$

Thus  $M[x]$  is only a quasi subvector space over  $F$ .

Clearly  $M[x]$  is an infinite dimensional over  $F$ .

$$W[x] = \left\{ \sum_{i=0}^5 a_i x^i \mid a_i \in [0, 34), 0 \leq i \leq 5, + \right\} \subseteq V;$$

$W[x]$  is a infinite dimensional vector subspace over  $F$  but  $W[x]$  is not a linear pseudo subalgebra over  $F$ .

$$\begin{aligned} D[x] &= \left\{ \sum_{i=0}^7 a_i x^i \mid a_i \in \{0, 0.5, 1, 1.5, 2, 2.5, \dots, 33.5\}, 0 \leq i \leq 7, \right. \\ &\quad \left. + \right\} \subseteq S[x] \end{aligned}$$

be a special quasi polynomial vector subspace over  $F$ .

Clearly  $D[x]$  is finite dimensional over  $F$ .

**Example 2.53:** Let

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in [0, 35), +, \times \right\}$$

is a special interval pseudo linear algebra over the field  $F = \{0, 7, 14, 21, 28\} \subseteq \mathbb{Z}_{35}$ .  $S[x]$  is infinite dimensional over  $F$ .  $S[x]$  has quasi subspaces both of finite and infinite dimensional.

Let

$$M[x] = \left\{ \sum_{i=0}^7 a_i x^i \mid a_i \in F, 0 \leq i \leq 7 \right\} \subseteq S[x]$$

be the quasi vector subspace over  $F$ . Clearly  $M[x]$  is not a linear pseudo subalgebra.

Let

$$W[x] = \left\{ \sum_{i=0}^5 a_i x^i \mid a_i \in F, 0 \leq i \leq 5 \right\} \subseteq S[x]$$

be the special linear pseudo subalgebra of  $V$  over  $F$ . Clearly dimension of  $W[x]$  is infinite over  $F$ .

$$B[x] = \left\{ \sum_{i=0}^8 a_i x^i \mid a_i \in \{0, 0.5, 1, 1.5, 2, 2.5, \dots, 34.5\} \right\}$$

$$\subseteq [0, 35), 0 \leq i \leq 8, + \} \subseteq V;$$

is a special quasi vector subspace of  $V$  of finite dimension over  $F$ . Clearly  $B[x]$  is not a linear pseudo subalgebra of  $V$  over  $F$ .

Infact  $V$  has infinitely many finite dimensional quasi vector subspaces over  $F$ . Also  $V$  has infinitely many infinite



dimensional quasi vector subspaces over  $F$  where none of them are linear pseudo subalgebras of  $V$  over  $F$ .

It is important to note none of the pseudo sublinear algebras are finite dimensional.

We see these polynomial pseudo linear algebra have sublinear pseudo algebras of infinite dimensions where at least one is of countable infinity.

**Example 2.54:** Let

$$V = S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in [0, 43), +, \times \right\}$$

be the special interval pseudo linear algebra over the field  $F = \mathbb{Z}_{43}$ .

Let

$$M_1 = \left\{ \sum_{i=5}^9 a_i x^i \mid a_i \in [0, 43), 5 \leq i \leq 9, +, \times \right\} \subseteq V$$

be a special interval quasi subspace over  $F$  and  $M_1$  is not a linear pseudo subalgebra of  $V$  over  $F$ .

For if  $p(x) = x^5 + 10x^8 + 20x^9$   
and  $q(x) = 6x^7 + 2x^6 \in M_1$

$$\begin{aligned} \text{Now } p(x) \times q(x) &= (x^5 + 10x^8 + 20x^9) \times (6x^7 + 2x^6) \\ &= 6x^{12} + 60x^{15} + 120x^{16} + 40x^{15} \\ &= 6x^{12} + 40x^{15} + 17x^{15} + 34x^{16} \\ &= 6x^{12} + 34x^{16} + 14x^{15} \notin M_1. \end{aligned}$$

As all polynomials in  $M_1$  is of degree less than or equal to 9 and greater than or equal to 5. Thus  $M_1$  can only be a quasi special subvector space of  $V$  over  $F$ .

$$N_1 = \left\{ \sum_{i=0}^7 a_i x^i \mid a_i \in [0, 43), +, 0 \leq i \leq 7 \right\} \subseteq V$$

be the special interval quasi vector subspace of  $V$  over  $F$ . Clearly  $N_1$  is infinite dimensional over  $F$  but  $N_1$  is not a pseudo linear subalgebra of  $V$  over  $F$ .

$$N_2 = \left\{ \sum_{i=0}^9 a_i x^i \mid a_i \in \mathbb{Z}_{43}, 0 \leq i \leq 9 \right\} \subseteq V$$

is a special quasi vector subspace of  $V$  over  $F$ . Clearly  $N_2$  is finite dimensional.

Further  $N_2$  is not a linear pseudo subalgebra of  $V$  over  $F$ .

$$N_3 = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in \{0, 0.5, 1, 1.5, \dots, 41, 41.5, 42, 42.5\}, \subseteq [0, 43), +, \times \right\} \subseteq V$$

is a special quasi vector subspace of  $V$  of infinite dimension over  $F$ .

Clearly  $N_3$  is not a linear pseudo subalgebras as  $p(x) = 9.5x^5 + 0.5$  and  $1.5x^8 + 0.5$  and  $q(x) = 1.5x^8 + 0.5 \in N_2$  then  $q(x) \times p(x) = (1.5x^8 + 0.5) \times (9.5x^5 + 0.5)$

$= 14.25x^{13} + 4.75x^5 + 0.75x^8 + 0.25 \notin N_3$  as none of the coefficients are in  $\{0, 0.5, 1, 1.5, 2, \dots, 42, 42.5\} \subseteq \{[0, 43)\}$ . Hence  $N_3$  is a linear pseudo subalgebra of  $V$  over  $F$ .

Now we proceed onto suggest some problems some of which are very difficult and can be realized as research problems.

## Problems

1. Obtain some special features enjoyed by special interval vector spaces.
2. Distinguish the properties between special interval linear algebras and vector spaces.
3. Give an example of a special interval vector space which is not a special interval linear algebra.
4. Show all special interval linear algebras are infinite dimensional.
5. Let  $V = \{[0, 21), +\}$  be the special interval vector space over the field  $F = \{0, 7, 14\} \subseteq \mathbb{Z}_{21}$ .
  - (i) Show  $V$  is infinite dimensional over  $F$ .
  - (ii) Find 5 subspaces of  $V$  of finite dimension over  $F$ .
  - (iii) Can  $V$  have subspaces of infinite dimension over  $F$ ?
  - (iv) Can  $V$  be written as a direct sum of subspaces?

6. Let  $V = \{[0, 47), +\}$  be the special interval vector space over the field  $\mathbb{Z}_{47}$ .

Study questions (i) to (iv) of problem 5 for this  $V$ .

7. Let  $V = \{[0, 29), +\}$  be the special interval vector space over the field  $\mathbb{Z}_{29}$ .

Study questions (i) to (iv) of problem 5 for this  $V$ .

8. Let  $V = \{[0, 6), +\}$  be the special interval vector space over  $F = \{0, 3\} \subseteq \mathbb{Z}_6$ .
  - (i) Study questions (i) to (iv) of problem 5 for this  $V$ .
  - (ii) Study questions (i) to (iv) of problem 5 for this  $V$  if  $F$  is replaced by the field  $\{0, 2, 4\} \subseteq \mathbb{Z}_6$ .

9. Let  $V = \{[0, 2p), +\}$ ,  $p$  a prime, be the special interval vector space over the field  $F = \{0, p\} \subseteq [0, 2p)$ .
  - (i) Study questions (i) to (iv) of problem 5 for this  $V$ .
  - (ii) Prove  $Z_{2p}$  has only two subsets which are fields.
10. Let  $V = \{[0, 24), +\}$  be a special interval vector space over a field  $F \subseteq Z_{24}$ .
  - (i) How many subsets in  $Z_{24}$  are fields?
  - (ii) Study questions (i) to (iv) of problem 5 for this  $V$ .
11. Let  $V = \{[0, Z_{30}), +\}$  be a special interval vector space over a field  $F = \{0, 10, 20\} \subseteq Z_{30}$ .
  - (i) Study questions (i) to (iv) of problem 5 for this  $V$ .
  - (ii) Find all subsets in  $Z_{30}$  which are fields of  $Z_{30}$ .
12. Prove all special interval vector spaces  $V = \{[0, n), +, n < \infty\}$  are always closed under  $\times \bmod n$  but product does not in general distribute over addition, that is  $a \times (b + c) \neq a \times b + a \times c$  for all  $a, b, c \in [0, n)$ . Hence  $V$  is a special interval linear algebra over  $F \subseteq Z_n$ .
  - (i) Prove  $V$  has quasi vector subspaces which are not special pseudo linear subalgebras over  $F$ .
13. Is it possible to have a special interval pseudo linear algebra which has no linear pseudo subalgebras?
14. Is it possible to have special interval linear pseudo algebra which has no special quasi vector subspaces?
15. Let  $V = \{[0, 28), +, \times\}$  be a special interval pseudo linear algebra over a field  $F \subseteq Z_{28}$ .
  - (i) Find special interval linear pseudo subalgebras of  $V$  over  $F$ .

- (ii) Can  $V$  have infinite number of linear pseudo subalgebras over  $F$ ?
- (iii) Is it possible to have linear pseudo subalgebras of  $V$  of finite dimension over  $F$ ?
- (iv) Can  $V$  have special quasi vector subspaces of infinite dimension over  $F$ ?
- (v) Is it possible for  $V$  to have special quasi vector subspaces of finite dimension over  $F$ ?
- (vi) How many infinite number of basis can  $V$  have?
- (vii) Find all subsets in  $Z_{28}$  which are subfields of  $Z_{28}$ .

16. Let  $V = \{[0, 48), +, \times\}$  be a special interval pseudo linear algebra over a field  $F \subseteq Z_{48}$ .

Study questions (i) to (vii) of problem 15 for this  $V$ .

17. Can  $Z_p$ ,  $p$  a prime be a  $S$ - pseudo special interval ring?

18. Let  $V = \{[0, 660), \times, +\}$  be a special interval pseudo linear algebra over  $F \subseteq Z_{660}$ .

Study questions (i) to (vii) of problem 15 for this  $V$ .

19. Let  $V = \{[0, 420), \times, +\}$  be a special interval pseudo linear algebra over  $F \subseteq Z_{420}$ .

Study questions (i) to (vii) of problem 15 for this  $V$ .

20. Let  $V = \{(a_1, a_2, a_3, a_4) \mid a_i \in [0, 83), 1 \leq i \leq 4, +, \times\}$  be the special interval pseudo linear algebra over  $F = Z_{83}$ .

- (i) Find all sublinear pseudo algebras of  $V$  over  $F$  of finite dimension.
- (ii) Find all quasi vector subspaces of  $V$  over  $F$  of finite and infinite dimension over  $F$ .
- (iii) Find all linear operators of  $V$ .
- (iv) Can  $V$  be written as a direct sum of sublinear pseudo algebras over  $F$ ?

(v) Find the algebraic structure enjoyed by

$$V_T = \{T : V \rightarrow V\}.$$

21. Let  $V = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in [0, 42), 1 \leq i \leq 5, +, \times\}$  the special interval pseudo linear algebra over  $F \subseteq Z_{42}$  ( $F$  a field in  $Z_{42}$ ).

- (i) Find all subsets of  $Z_{42}$  which are fields in  $Z_{42}$ .
- (ii) Study questions (i) to (vii) of problem 15 for this  $V$  over all the fields in  $Z_{42}$ .

22. Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_7 \end{bmatrix} \mid a_i \in [0, 19; 1 \leq i \leq 7, +, \times_n \right\}$$

be the special interval pseudo linear algebra over  $F = Z_{19}$ .

- (i) Study questions (i) to (vii) of problem 15 for this  $V$  over all the fields in  $Z_{42}$ .

23. Let

$V = \{(a_1 \mid a_2 \ a_3 \ a_4 \mid a_5 \ a_6 \mid a_7) \mid a_i \in [0, 46), 1 \leq i \leq 7, +, \times\}$  be the special interval pseudo linear algebra over a field  $F \subseteq Z_{46}$ .

- (i) Study questions (i) to (vii) of problem 15 for this  $V$ .
- (ii) Find all subspaces (sublinear pseudo algebras) which are orthogonal to  $P = \{(a_1 \mid 0 \ 0 \ 0 \mid a_2 \ 0 \mid a_3) \mid a_i \in [0, 46), 1 \leq i \leq 3, +, \times\} \subseteq V$ .

24. Let  $V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix} \mid a_i \in [0, 86), 1 \leq i \leq 10, +, \times_n \right\}$  be the

special interval pseudo algebra over a field  $\{0, 43\} \subseteq \mathbb{Z}_{86}$ .

Study questions (i) to (vii) of problem 15 for this  $V$ .

25. Let

$$V = \left\{ \begin{bmatrix} \overline{a_1} \\ \overline{a_2} \\ \overline{a_3} \\ \overline{a_4} \\ \overline{a_5} \\ \overline{a_6} \\ \overline{a_7} \end{bmatrix} \mid a_i \in [0, 22), 1 \leq i \leq 7, +, \times_n \right\}$$

be the special pseudo interval algebra over a field  $F \subseteq \mathbb{Z}_{22}$ .

Study questions (i) to (vii) of problem 15 for this  $V$ .

26. Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & \dots & \dots & \dots & \dots & a_{12} \\ a_{13} & \dots & \dots & \dots & \dots & a_{18} \\ a_{19} & \dots & \dots & \dots & \dots & a_{24} \\ a_{25} & \dots & \dots & \dots & \dots & a_{30} \\ a_{31} & \dots & \dots & \dots & \dots & a_{36} \end{bmatrix} \mid a_i \in [0, 61), 1 \leq i \leq 36, \right. \\ \left. +, \times \right\}$$

be the special pseudo interval algebra over a field  $F = \mathbb{Z}_{61}$ . Clearly  $V$  is a non commutative pseudo linear algebra over  $F$ .

- (i) Study questions (i) to (vii) of problem 15 for this  $V$ .
- (ii) Find all commutative linear pseudo subalgebras and commutative vector subspaces of  $V$  over  $F = \mathbb{Z}_{61}$ .

27. Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \mid a_i \in [0, 11), 1 \leq i \leq 16, +, \times \right\}$$

be the special interval linear pseudo algebra over  $F = \mathbb{Z}_{11}$ .

- (i) Study questions (i) to (vii) of problem 15 for this  $V$ .
- (ii) When are these sublinear pseudo algebras Commutative?
- (iii) Find those subvector spaces which are commutative.

28. Let

$$V = \left\{ \left[ \begin{array}{cc|cc|c} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & \dots & \dots & \dots & \dots & a_{12} \\ \hline a_{13} & \dots & \dots & \dots & \dots & a_{18} \\ a_{19} & \dots & \dots & \dots & \dots & a_{24} \\ \hline a_{25} & \dots & \dots & \dots & \dots & a_{30} \\ a_{31} & \dots & \dots & \dots & \dots & a_{36} \end{array} \right] \mid a_i \in [0, 23), 1 \leq i \leq 36, +, \times \right\}$$

is a special interval linear pseudo algebra over  $F = \mathbb{Z}_{23}$ .  
Clearly  $V$  is non commutative.

- (i) Study questions (i) to (vii) of problem 15 for this  $V$ .
- (ii) Study questions (ii) of problem 26 for this  $S$ .



29. Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_5 \\ a_6 & a_7 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{15} \\ a_{16} & a_{17} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{25} \end{bmatrix} \mid a_i \in [0, 15), 1 \leq i \leq 25, +, \times_n \right\}$$

be the special interval pseudo linear algebra over the field  $F \subseteq Z_{15}$  ( $F \neq \emptyset$  a subset which is a field).

(i) Study questions (i) to (vii) of problem 15 for this  $V$ .

30. Prove all special interval polynomial linear pseudo algebras are always of infinite dimension (if  $x^n = 1$  is not used for  $n < \infty$ ) over the field on which it is defined.

31. Obtain some special and interesting features enjoyed by special interval polynomial linear pseudo algebra defined over a field  $F$ .

32. Let

$$V = \left\{ \left[ \begin{array}{cc|cc|c|c} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & \dots & \dots & \dots & \dots & a_{12} \\ \hline a_{13} & \dots & \dots & \dots & \dots & a_{18} \\ a_{19} & \dots & \dots & \dots & \dots & a_{24} \\ \hline a_{25} & \dots & \dots & \dots & \dots & a_{30} \end{array} \right] \mid a_i \in [0, 43), 1 \leq i \leq 30, +, \times_n \right\}$$

be the special interval linear pseudo algebra over the field  $Z_{43}$ .

Study questions (i) to (vii) of problem 15 for this  $V$ .

33. Let

$$V = \left\{ \left( \begin{array}{cc|cc|cc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \end{array} \right) \mid a_i \in [0, 241), \right. \\ \left. 1 \leq i \leq 18, +, \times_n \right\}$$

be the special interval linear pseudo algebra over the field  $Z_{241}$ .

Study questions (i) to (vii) of problem 15 for this  $V$ .

34. Let

$$V = \left\{ \left[ \begin{array}{cc} a_1 & a_2 \\ a_3 & a_4 \end{array} \right] \mid a_i \in [0, 3), 1 \leq i \leq 4, +, \times \right\}$$

be the special interval linear pseudo algebra over the field  $F = Z_3$ .

Study questions (i) to (vii) of problem 15 for this  $V$ .

35. Give special features enjoyed by special pseudo interval linear algebras.

36. Let

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_j \in [0, 7), +, \times \right\}$$

be the special interval polynomial linear pseudo algebra over the field  $Z_7 = F$ .

(i) Study the special linear pseudo subalgebras of  $V$  over  $F$ .

- (ii) Will every sublinear pseudo algebra of  $V$  over  $F$  be infinite dimension?
- (iii) Find quasi vector subspaces of  $V$  over  $F$  of both finite and infinite dimension over  $F$ .

37. Let

$$V = \left\{ \sum_{i=0}^{19} a_i x^i \mid a_j \in [0, 46), 0 \leq i \leq 19, + \right\}$$

be a special interval vector space over the field  $F = \{0, 23\} \subseteq \mathbb{Z}_{46}$ .

- (i) Show  $V$  is not a special linear pseudo algebra.
- (ii) Prove dimension of  $V$  over  $F$  is infinite.
- (iii) Find vector subspaces of  $V$  which are finite dimensional over  $F$ .
- (iv) How many vector subspaces of  $V$  over  $F$  are finite dimensional?
- (v) Find all vector subspaces of  $V$  over  $F$  of infinite dimension.
- (vi) Can subspaces of  $V$  of finite dimension have more than one basis?
- (vii) Is it possible to have a vector subspace of dimension 8?

38. Let

$$V = \left\{ \sum_{i=0}^{90} a_i x^i \mid a_j \in [0, 5), 0 \leq i \leq 90, + \right\}$$

be a special interval vector space over  $F = \mathbb{Z}_5$ .

Study questions (i) to (vii) of problem 37 for this  $V$ .

- 39. What will happen if  $[0, 5)$  in problem 38 is replaced by  $[0, 51)$ ?

Study questions (i) to (vii) of problem 37 for this  $V$  with  $[0, 5)$  replaced by  $[0, 51)$ .

## Chapter Three

# SMARANDACHE SPECIAL INTERVAL PSEUDO LINEAR ALGEBRAS

In this chapter we define two new concepts viz., Smaradache special interval pseudo linear algebra (S-special interval pseudo linear algebra) and Smarandache strong special interval pseudo linear algebra. They are illustrated by examples and described and developed in this chapter.

**DEFINITION 3.1:** *Let  $S = \{[0, n), +\}$  be the additive abelian group. Let  $Z_n \subseteq [0, n)$  be Smarandache ring. Let  $S$  be a special interval vector space we define  $S$  as a Smarandache special interval vector space over the  $S$ -ring  $Z_n$ .*

Here instead of the field in  $Z_n$  we use the totality of  $Z_n$ . We give examples of them.

**Example 3.1:** Let  $S = \{[0, 6), +\}$  be a S-special interval vector space over the S-ring  $Z_6$ .

**Example 3.2:** Let  $S = \{[0, 15), +\}$  be a S-special interval vector space over a ring S-ring  $Z_{15}$ .

**Example 3.3:** Let  $S = \{[0, 14), +\}$  be a S-special interval vector space over the S-ring  $Z_{14}$ .

**Example 3.4:** Let  $S = \{[0, 46), +\}$  be a S-special interval vector space over the S-ring  $Z_{46}$ .

**Example 3.5:** Let  $S = \{[0, 21), +\}$  be the S-special interval vector space over the S-ring  $Z_{21}$ .

We prove the following theorems.

**THEOREM 3.1:** Let  $S = \{[0, 2p), +\}$ ,  $p$  a prime;  $S$  is a S-special interval vector space over the S-ring  $Z_{2p}$ .

Proof follows from the simple fact as  $Z_{2p}$  is a S-ring for  $\{0, p\} \subseteq Z_{2p}$  is a field.

**THEOREM 3.2:**  $S = \{[0, 3p), +\}$  ( $p$  a prime) is a S-special interval vector space over the S-ring  $Z_{3p}$ .

Proof follows from the fact  $Z_{3p}$  is a S-ring. Hence the claim.

**THEOREM 3.3:**  $S = \{[0, pq), +\}$  ( $p$  and  $q$  are primes) is a S-special interval vector space over the S-ring  $Z_{pq}$ .

**Example 3.6:** Let  $S = \{[0, 33), +\}$  be the S-special interval vector space over the S-ring  $Z_{33}$ .

We will describe vector subspaces of  $S$  over the S-ring.

**Example 3.7:** Let  $S = \{[0, 14), +\}$  be a special interval vector space over the S-ring  $Z_{14}$ .  $S$  is infinite dimensional over  $Z_{14}$ .

Let

$P = \{\{0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, \dots, 11.5, 12, 12.5, 13, 13.5\}, +\}$  be a S-special interval vector subspace of  $S$  over the S-ring  $Z_{14}$ . Clearly  $P$  is finite dimensional over the S-ring  $Z_{14}$ .

$M = \{\{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, \dots, 13, 13.2, 13.4, 13.6, 13.8\}, +\} \subseteq S$  is again a finite dimensional vector subspace of  $S$  over  $Z_{14}$ . We have infinite number of finite dimensional vector subspaces over  $Z_{14}$ .

$T = \{\{0, 0.1, 0.2, \dots, 0.9, 1, 1.1, 1.2, \dots, 2, 2.1, 2.2, \dots, 13, 13.1, 13.2, \dots, 13.9\}, +\} \subseteq S$  be the finite dimensional vector subspace of  $V$  over the  $S$ -ring  $Z_{14}$ .

**Example 3.8:** Let  $S = \{[0, 33), +\}$  be a  $S$ -special interval vector space of  $V$  over the  $S$ -ring  $R = Z_{33}$ .

$M_1 = \{\{0, 1, 2, \dots, 32), +\} \subseteq V$  be the  $S$ -special vector subspace of  $V$  of dimension 1.

$M_2 = \{\{0, 0.5, 1, 1.5, 2, 2.5, \dots, 31, 31.5, 32, 32.5\}, +\} \subseteq V$  be the  $S$ -special interval vector subspace of  $V$  over the  $S$ -ring  $R$  of finite dimension.

$M_3 = \{\{0, 0.1, 0.2, \dots, 0.9, 1, 1.1, 1.2, \dots, 1.9, 2, 2.1, \dots, 2.9, \dots, 32.1, 32.2, \dots, 32.9\}, +\} \subseteq V$  be the  $S$ -special interval vector subspace of  $V$  over the  $S$ -ring  $R$ .

Clearly  $M_3$  is finite dimensional vector subspace of  $V$  over  $R$ .

$M_4 = \{\{0, 0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75, 2, \dots, 32, 32.25, 32.5, 32.75\}, +\} \subseteq V$  be the  $S$ -special interval vector subspace of  $V$  over the  $S$ -ring  $R$ .

Thus  $V$  has several finite dimensional  $S$ -special interval vector subspaces of  $V$  over  $R$ .

**Example 3.9:** Let  $V = \{[0, 35), +\}$  be the  $S$ -special interval vector space over the  $S$ -ring  $R = Z_{35}$ .

$T_1 = \{\{0, 1, 2, \dots, 34\}, +\} \subseteq V$  be subvector space of  $V$  over  $R$ .

$T_2 = \{\{0, 5, 10, 15, 20, 25, 30\}, +\} \subseteq V$  be the vector subspace of  $V$  over  $R$ .

Both  $T_1$  and  $T_2$  are finite dimensional over  $R$ .

$T_3 = \{\{0, 7, 14, 21, 28\}, +\} \subseteq V$  be the vector subspace of  $V$  over  $R$ .  $T_3$  is also finite dimensional over  $Z_{35}$ .

$T_4 = \{\{0, 0.1, 0.2, 0.3, \dots, 0.9, 1, 1.1, 1.2, \dots, 2, \dots, 30, 3.1, 3.2, \dots, 30.9, \dots, 34.1, 34.2, \dots, 34.9\}, +\} \subseteq V$  is also a vector subspace of  $V$  over  $R = Z_{35}$ .

$T_4$  is also finite dimensional over  $Z_{35}$  and so on.

**Example 3.10:** Let  $V = \{[0, 28), +\}$  be the S-special interval vector space over the S-ring  $R = Z_{28}$ .

$P_1 = \{\{0, 2, 4, 6, 7, \dots, 26\}, +\} \subseteq V$  is a S-vector subspace of  $V$  of finite dimension over  $R = Z_{28}$ .

$P_2 = \{\{0, 4, 8, 12, 16, 20, 24\}, +\} \subseteq V$  is also a S-vector subspace of  $V$  over  $R = Z_{28}$ .

$P_3 = \{\{0, 7, 14, 21\}, +\} \subseteq V$  is a S-vector subspace of  $V$  over  $R = Z_{28}$ .

$P_4 = \{\{0, 14\}, +\} \subseteq V$  is a S-vector subspace of  $V$  over  $R = Z_{28}$ .

All the four subspaces are of finite dimension over  $Z_{28}$ .

$T_1 = \{\{0, 0.5, 1, 1.5, 2, 2.5, 3, \dots, 27, 27.5\}, +\} \subseteq V$  is a S-vector subspace of  $V$  over  $R = Z_{28}$ .

$T_2 = \{\{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, 1.6, 1.8, 2, \dots, 27, 27.2, 27.4, 27.6, 27.8\}, +\} \subseteq V$  is a S-vector subspace of  $V$  over the S-ring  $R = Z_{28}$ .

**Example 3.11:** Let  $V = \{[0, 26), +\}$  be the S-special interval vector space over the S-ring  $R = Z_{26}$ .

This has S-subspaces of both finite and infinite dimension.

If on the S-special interval vector space  $V$  over the S-ring we can define a compatible product  $\times$  on  $V$  then we define  $V$  to be a S-special interval pseudo linear algebra over the S-ring  $Z_n$ , since  $a \times (b + c) \neq a \times b + a \times c$  for all  $a, b, c \in V$ .

We see  $V$  has substructure which are not linear subalgebras only quasi vector subspaces of  $V$  over the S-ring  $Z_n$ .

We will illustrate this situation by some examples.

**Example 3.12:** Let  $V = \{[0, 22), +, \times\}$  be a S-special pseudo interval linear algebra over the S-ring  $Z_{22} = R$ . Clearly  $V$  has S-quasi vector subspaces given by

$P_1 = \{\{0, 1, 2, \dots, 21\}, +, \times\} \subseteq V$  is a S-special interval linear pseudo subalgebra of  $V$  over  $R$ .

$P_2 = \{\{0, 0.5, 1.0, 1.5, 2, 2.5, \dots, 20, 20.5, 21, 21.5\}, +, \times\} \subseteq V$  is a S-special quasi vector subspace of  $V$  over  $R = Z_{22}$ .

$P_3 = \{\{0, 0.1, 0.2, \dots, 0.9, 1, 1.1, 1.2, \dots, 1.9, 2, \dots, 20, 20.1, \dots, 21, 21.1, 21.2, \dots, 21.9\}, +, \times\} \subseteq V$  is a S-special quasi vector subspace of  $V$  over  $R = Z_{22}$ . Thus we have several S-special quasi vector subspaces of  $V$  over  $R = Z_{22}$ .

Now  $P_4 = \{\{0, 2, 4, 6, \dots, 20\}, +, \times\} \subseteq V$  is a S-special linear pseudo subalgebra of  $V$  over  $R = Z_{22}$ .

Similarly  $P_5 = \{\{0, 11\}, +, \times\} \subseteq V$  is a S-special pseudo linear subalgebra of  $V$  over  $R = Z_{22}$ .

**Example 3.13:** Let  $V = \{[0, 24), +, \times\}$  be a S-special pseudo linear algebra over the S-ring  $R = Z_{24}$ .  $V$  has finite dimensional



S-quasi vector subspaces of finite dimension as well as finite dimensional S-special pseudo linear subalgebras.

Let  $M_1 = \{\{0, 0.5, 0.15, 2, \dots, 23, 23.5\}, +, \times\} \subseteq V$  be a S-quasi vector subspace of  $V$  over  $R$  of finite dimension over  $R$ .

$M_2 = \{\{0, 0.2, 0.4, \dots, 1, 1.2, 1.4, \dots, 2, 2.2, 2.4, \dots, 3, 23, 23.2, \dots, 23.8\}, +, \times\} \subseteq V$  is a S-quasi subvector space of  $V$  over the S-ring  $Z_{24}$ .

$M_3 = \{\{0, 1, 2, \dots, 23\}, +, \times\} \subseteq V$  is a S-special pseudo linear subalgebra of  $V$  over the S-ring  $Z_{24}$ .

$M_4 = \{\{0, 2, 4, 6, 8, \dots, 22\}, +, \times\} \subseteq V$  is a S-special pseudo linear subalgebra of  $V$  over the S-ring  $Z_{24}$ .

$M_5 = \{\{0, 3, 6, 9, 12, 15, 18, 21\}, +, \times\} \subseteq V$  is a S-special linear pseudo subalgebra of  $V$  over the S-ring  $Z_{24}$ .

$M_6 = \{\{0, 4, 8, 12, 16, 20\}, +, \times\} \subseteq V$  is a S-special pseudo linear subalgebra of  $V$  over the S-ring  $Z_{24}$ .

All these S-special quasi vector subspaces as well as all the S-special linear pseudo subalgebra of  $V$  over the ring  $R$ .

**Example 3.14:** Let  $V = \{[0, 105), +, \times\}$  be the S-special pseudo linear algebra over the S-ring  $R = Z_{105}$ .  $V$  has S-special quasi vector subspaces of finite dimension one  $R = Z_{105}$  which is as follows:

$N_1 = \{\{0, 0.5, 1, 1.5, \dots, 104, 104.5\}, +, \times\} \subseteq V$  is only a S-special quasi vector subspace of  $V$  over  $R$ . Clearly  $N_1$  is of finite dimension over  $R$ .

$N_2 = \{\{0, 1, 2, \dots, 105\}, +, \times\} \subseteq V$  is a S-special pseudo linear subalgebra of  $V$  over  $R = Z_{105}$ .

Clearly dimension of  $N_2$  over  $R$  is one.

$N_3 = \{\{0, 5, 10, 15, 20, \dots, 100\}, +, \times\} \subseteq V$  is a S-special linear subalgebra of  $V$  over  $R = Z_{105}$ . Clearly  $N_3$  over  $R$  is finite dimensional.

$N_4 = \{\{0, 3, 6, 9, 12, 15, \dots, 102\}, +, \times\} \subseteq V$  is a S-special linear subalgebra of  $V$  over  $R$ .

$N_5 = \{\{0, 0.1, 0.2, \dots, 0.9, 1, 1.1, 1.2, \dots, 1.9, 2, \dots, 104, 104.1, 104.2, \dots, 104.9\}, +, \times\} \subseteq V$  is a S-special quasi vector subspace of  $V$  over  $R = Z_{105}$ .

**Example 3.15:** Let

$V = \{(a_1, a_2, a_3, a_4) \mid a_i \in [0, 22), 1 \leq i \leq 4, +\}$  be a S-special interval vector space over the S-ring  $R = Z_{22}$ .  $V$  has several S-special interval vector subspaces some of which are finite dimensional are some of them are infinite dimensional over S-ring  $R = Z_{22}$ .

$M_1 = \{(a_1, 0, 0, 0) \mid a_1 \in [0, 22), +\} \subseteq V$  is a S-special interval vector subspace of  $V$  over the S-ring  $R = Z_{22}$ .

$M_2 = \{(0, a_2, 0, 0) \mid a_2 \in [0, 22), +\} \subseteq V$  be the S-special interval vector subspace of  $V$  over the S-ring  $R = Z_{22}$ .

$M_3 = \{(0, 0, a_3, 0) \mid a_3 \in [0, 22), +\} \subseteq V$  be the S-special interval vector subspace of  $V$  over the S-ring  $R = Z_{22}$ .

$M_4 = \{(0, 0, 0, a_4) \mid a_4 \in [0, 22), +\} \subseteq V$  be the S-special interval vector subspace of  $V$  over the S-ring  $R = Z_{22}$ .

Clearly  $V = M_1 + M_2 + M_3 + M_4$  is a direct sum and  $M_i \cap M_j = \{(0, 0, 0, 0)\}; i \neq j; 1 \leq i, j \leq 4$ .

All of four spaces are infinite dimensional over  $R = Z_{22}$ .

Let  $T_1 = \{(a_1, 0, 0, 0) \mid a_1 \in \{0, 0.5, 1, 1.5, 2, 2.5, \dots, 20, 20.5, 21, 21.5\}, +\} \subseteq V$  be a S-special subspace of  $V$  over the S-ring  $R = Z_{22}$  of finite dimension over  $R = Z_{22}$ .

Let  $P_1 = \{(a_1, 0, 0, a_2) \mid a_2 \in [0, 22), a_1 \in \{0, 0.1, 0.2, \dots, 0.9, 1, 1.1, \dots, 20, 20.1, \dots, 20.9, 21, 21.1, \dots, 21.9\} \subseteq [0, 22), +\} \subseteq V$  be a S-special interval subspace of  $V$  over the S-ring  $R = Z_{22}$ .

We see  $P_2 = \{(0, a_1, a_2, 0) \mid a_1, a_2 \in [0, 22), +\} \subseteq V$  is a special vector subspace of  $V$  over the S-ring  $Z_{22}$ .

$P_3 = \{(0, 0, a_1, a_2) \mid a_1 \in Z_2, a_2 = \{(0, 0.1, 0.2, \dots, 1, 1.1, 1.2, \dots, 20, 20.1, \dots, 20.9, 21, \dots, 21.9\} \subseteq [0, 22), +\} \subseteq V$  is a S-special vector subspace of  $V$  over the S-ring  $Z_{22}$ .

All these subspaces  $P_3, P_1$  and  $T_1$  cannot be made into a S-special pseudo linear subalgebras only a S-special vector subspace of  $V$  over the S-ring  $Z_{22}$ .

**Example 3.16:** Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \mid a_i \in [0, 24), 1 \leq i \leq 6, +, \times_n \right\}$$

is a S-special pseudo linear algebra over the S-ring  $Z_{24}$ .

$V$  has several S-special pseudo sublinear algebras.

$$T_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_1 \in [0, 24), +, \times_n \right\} \subseteq V.$$

$$T_2 = \left\{ \begin{bmatrix} 0 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| a_2 \in [0, 24), +, \times_n \} \subseteq V,$$

$$T_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ a_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| a_3 \in [0, 24), +, \times_n \} \subseteq V,$$

$$T_4 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_4 \\ 0 \\ 0 \end{bmatrix} \middle| a_4 \in [0, 24), +, \times_n \} \subseteq V,$$

$$T_5 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_5 \\ 0 \end{bmatrix} \middle| a_5 \in [0, 24), +, \times_n \} \subseteq V \text{ and}$$

$$T_6 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_6 \end{bmatrix} \mid a_6 \in [0, 24), +, \times_n \} \subseteq V$$

are S-special interval pseudo sublinear algebras of  $V$  over  $F$ .

$$\text{Clearly } T_i \cap T_j = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ if } i \neq j, 1 \leq i, j \leq 6.$$

Further  $V = T_1 + T_2 + T_3 + T_4 + T_5 + T_6$  is a direct sum of sublinear pseudo algebras of  $V$  over  $R = Z_{24}$ .

Let

$$P_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_1 \in \{0, 0.1, 0.2, \dots, 0.9, 1, 1.1, 1.2, \dots, 23, 23.1, \dots, 23.9\} \subseteq [0, 24); + \} \subseteq V$$

is only a S-special quasi vector subspace of  $V$  and not a S-pseudo linear subalgebra of  $V$  over the S-ring  $R$ .

We have several such S-special quasi vector subspaces which are finite dimensional over R.

We see even in case of S-special interval pseudo linear algebras we have S-subalgebras or S-quasi subvector spaces of V such that they are orthogonal with each other.

Let

$$B_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_1, a_2 \in [0, 24), +, \times_n \} \subseteq V$$

be the S-special pseudo linear subalgebra of V.

$$B_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \end{bmatrix} \mid a_1, a_2 \in \{0, 0.5, 1, 1.5, \dots, 23, 23.5, \subseteq V, +\} \subseteq V$$

be the S-special quasi vector subspace of V over the S-ring  $R = Z_{24}$ .

$$\text{We see } B_1 \times B_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

That is  $B_1$  is orthogonal to  $B_2$  and vice versa. However  $B_2$  is finite dimensional where as  $B_1$  is infinite dimensional  $B_1$  is a S-sublinear pseudo algebra but  $B_2$  is a S-quasi vector subspace of  $V$ .

Also  $B_1 + B_2 \neq V$  only a special quasi vector subspace of  $V$  over  $R$ .

$$\text{Infact take } B_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \mid a_i \in \{0, 0.5, 1, 1.5, \dots, 23, 23.5\} \subseteq \right.$$

$[0, 24); 1 \leq i \leq 4, +\} \subseteq V$  is only a S-special quasi vector subspace of  $V$  over the S-ring  $R = Z_{24}$ .

$$\text{We see if } x = \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ 0 \\ 0.5 \\ 1.5 \end{bmatrix} \text{ and } y = \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ 1.5 \\ 1.5 \\ 0.5 \end{bmatrix} \text{ are in } B_3;$$

$$\text{then } x \times_n y = \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ 0 \\ 0.5 \\ 1.5 \end{bmatrix} \times_n \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ 1.5 \\ 1.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.25 \\ 0 \\ 0.75 \\ 0.75 \end{bmatrix} \notin B_3.$$

Thus  $B_3$  is only a S-special quasi vector subspace of  $V$  over  $R$ . Further  $B_3$  is orthogonal with  $B_1$  but  $B_3$  is not orthogonal with  $B_2$ .

$B_1$  is also orthogonal with  $B_3$ ;  $B_3$  only a S-special quasi vector subspace of  $V$  over  $R$ .

**Example 3.17:** Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \mid a_i \in [0, 28); 1 \leq i \leq 9, +, \times \right\}$$

be a S-special interval pseudo linear algebra over the S-ring  $R = Z_{28}$ . We have S-quasi special vector subspaces of finite dimension over  $Z_{28}$ . However  $V$  is a non commutative S-special pseudo interval linear algebra.

Take

$$M_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid a_i \in \{0, 0.5, 1, 1.5, 2, \dots, 27, 27.5\} \right. \\ \left. \subseteq Z_{28} \right\} \subseteq V$$

is only a S-special quasi vector subspace of  $V$  over the S-ring  $Z_{28}$ . The dimension of  $M_1$  over  $V$  is finite dimensional over  $R = Z_{28}$ .



Let

$$M_2 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ a_4 & a_5 & a_6 \end{bmatrix} \mid a_i \in [0, 28), 1 \leq i \leq 3 \text{ and } a_j \in \right.$$

$$\{0, 0.1, 0.2, 0.3, \dots, 1, 1.1, \dots, 2, 27.1, 27.2, \dots, 27.9\} \subseteq V,$$

$$4 \leq j \leq 6; +, \times \} \subseteq V;$$

$M_2$  is a S-special interval quasi vector subspace of  $V$  over  $R = Z_{28}$ . Clearly  $M_2$  is a infinite dimensional subspace of  $V$  over  $Z_{28} = R$ .

We see for  $B, A \in M_2$  in general  $A \times B \notin M_2$ .

We show  $V$  is a S-interval non commutative special interval pseudo linear algebra.

$$\text{Let } A = \begin{bmatrix} 0.5 & 3 & 4 \\ 1 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 0.5 \\ 0 & 1 & 2 \end{bmatrix} \in M;$$

$$A \times B = \begin{bmatrix} 0.5 & 3 & 4 \\ 1 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 0.5 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0.5 & 17 & 9.5 \\ 1 & 10 & 1 \\ 0 & 8 & 8.5 \end{bmatrix} \dots I$$

Consider

$$B \times A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 0.5 \\ 0 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} 0.5 & 3 & 4 \\ 1 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2.5 & 7 & 4 \\ 4 & 8.5 & 2 \\ 1 & 4 & 8 \end{bmatrix} \dots \text{II.}$$

Clearly I and II are distinct. So  $V$  is a  $S$ -special interval non commutative pseudo linear algebra under usual matrix multiplication.

$$\text{Let } T_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \mid a_i \in \{0, 0.5, 1, 1.5, 2, 2.5, \dots, 27, \right.$$

$27.5\} \subseteq V; 1 \leq i \leq 3; +, \times\} \subseteq V$  be a  $S$ -special interval quasi vector subspace of  $V$  and is not a pseudo linear algebra.

Further  $T_1$  is finite dimensional subspace of  $V$  over  $F = Z_{28}$ .

Let

$$A = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 2.5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 6.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \in T_1.$$

$$A \times B = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 2.5 \end{bmatrix} \times \begin{bmatrix} 6.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$= \begin{bmatrix} 3.25 & 0 & 0 \\ 0 & 0.75 & 0 \\ 0 & 0 & 3.75 \end{bmatrix} \notin T_1.$$

Thus  $T_1$  is only a finite dimensional  $S$ -subspace.

**Example 3.18:** Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \mid a_i \in [0, 6); 1 \leq i \leq 9, +, \times_n \right\}$$

be the S-special interval pseudo linear algebra over the ring  $R = \mathbb{Z}_6$ .

$V$  is a commutative S-special interval pseudo linear algebra of infinite order.

Let

$$M_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 0 \end{bmatrix} \mid a_i \in [0, 6); 1 \leq i \leq 6, +, \times_n \right\} \subseteq V;$$

$M_1$  is a S-special interval pseudo linear subalgebra of  $V$  over  $R = \mathbb{Z}_6$ .

$$M_2 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \mid a_i \in \{0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5, 5.5\} \subseteq \mathbb{Z}_6; 1 \leq i \leq 3, +, \times_n \right\} \subseteq V$$

be the S-special interval quasi vector subspace of  $V$  over  $F = \mathbb{Z}_6$ .

Clearly  $M_2$  is not a S-special pseudo linear algebra is only a S-special quasi vector subspace over  $V$ .

Let  $A, B \in M_2$  where

$$A = \begin{bmatrix} 0.5 & 0 & 0 \\ 2.5 & 0 & 0 \\ 0 & 0 & 3.5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \in M_2.$$

We find

$$\begin{aligned} A \times_n B &= \begin{bmatrix} 0.5 & 0 & 0 \\ 2.5 & 0 & 0 \\ 0 & 0 & 3.5 \end{bmatrix} \times_n \begin{bmatrix} 0.5 & 0 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \\ &= \begin{bmatrix} 0.25 & 0 & 0 \\ 1.25 & 0 & 0 \\ 0 & 0 & 1.75 \end{bmatrix} \notin M_2. \end{aligned}$$

Clearly  $M_2$  is not a  $S$ -linear pseudo subalgebra only a  $S$ -quasi vector subspace of  $V$  over  $R = Z_6$ .  $M_2$  is commutative and finite dimensional over  $R = Z_6$ .

$$\text{Let } M_3 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ a_3 & 0 & 0 \end{bmatrix} \mid a_i \in \{0, 0.1, 0.2, \dots, 0.9, 1, 1.1, \right.$$

$1.2, \dots, 5, 5.1, 5.2, \dots, 5.9\} \subseteq Z_6, 1 \leq i \leq 3, +, \times_n \} \subseteq V$  is again a  $S$ -special quasi vector subspace of  $V$  over the  $S$ -ring  $Z_6$ .

Let

$$A = \begin{bmatrix} 0.2 & 0 & 0 \\ 0.1 & 0 & 0 \\ 0.9 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2.2 & 0 & 0 \\ 5.8 & 0 & 0 \\ 6.4 & 0 & 0 \end{bmatrix} \in M_3;$$

$$\text{we see } A \times_n B = \begin{bmatrix} 0.2 & 0 & 0 \\ 0.1 & 0 & 0 \\ 0.9 & 0 & 0 \end{bmatrix} \times_n \begin{bmatrix} 2.2 & 0 & 0 \\ 5.8 & 0 & 0 \\ 6.4 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.44 & 0 & 0 \\ 0.58 & 0 & 0 \\ 5.76 & 0 & 0 \end{bmatrix} \notin M_3.$$

So  $M_3$  is only a S-special interval quasi vector subspace of  $V$  over  $R = Z_6$ .

Clearly  $M_3$  is finite dimensional over  $R = Z_6$  however  $M_3$  is not a special interval pseudo linear subalgebra over  $Z_6$ .

Let

$$M_4 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_1 & 0 \\ a_2 & 0 & a_3 \end{bmatrix} \mid a_i \in [0, 6), 1 \leq i \leq 3 \right\} \subseteq V$$

be the S-special interval pseudo linear algebra over the S-ring  $Z_6$ .  $M_4$  is also infinite dimensional over the S-ring  $R = Z_6$ .

Consider

$$P_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid a_1 \in [0, 6), +, \times_n \right\} \subseteq V,$$

$$P_2 = \left\{ \begin{bmatrix} 0 & a_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid a_2 \in [0, 6), +, \times_n \right\} \subseteq V,$$

$$P_3 = \left\{ \begin{bmatrix} 0 & 0 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid a_3 \in [0, 6), +, \times_n \right\} \subseteq V,$$

$$P_4 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ a_4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| a_4 \in [0, 6), +, \times_n \subseteq V, \right.$$

$$P_5 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| a_5 \in [0, 6), +, \times_n \subseteq V, \dots, \text{ and} \right.$$

$$P_9 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_9 \end{bmatrix} \middle| a_9 \in [0, 6), +, \times_n \subseteq V \right.$$

are the nine S-special interval pseudo linear subalgebras of  $V$  over  $R = Z_6$ .

$$\text{We see } P_i \cap P_j = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \text{ if } i \neq j, 1 \leq i, j \leq 9.$$

However  $V = B_1 + B_2 + B_3 + B_4 + B_5 + B_6 \subseteq V$ .

Hence is not a direct sum of pseudo sublinear algebras over  $Z_6$ .

**Example 3.19:** Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix} \middle| a_i \in [0, 12), 1 \leq i \leq 30, +, \times_n \right\}$$

be the S-special pseudo linear algebra over the S-ring  $Z_{12}$ .

$V$  has both quasi vector subspaces and pseudo sublinear algebras of finite dimension as well as infinite dimension over the S-ring  $R = Z_{12}$ .

Let

$$A_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \middle| a_1 \in [0, 12), +, \times_n \subseteq V, \right.$$

$$A_2 = \left\{ \begin{bmatrix} 0 & a_2 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \middle| a_2 \in [0, 12), +, \times_n \subseteq V, \right.$$

$$A_3 = \left\{ \begin{bmatrix} 0 & 0 & a_3 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \middle| a_3 \in [0, 12), +, \times_n \subseteq V \text{ and so on.} \right.$$

$$A_{27} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & a_{27} \\ 0 & 0 & 0 \end{bmatrix} \middle| a_{27} \in [0, 12), +, \times_n \subseteq V, \right.$$

$$A_{28} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ a_{28} & 0 & 0 \end{bmatrix} \middle| a_{28} \in [0, 12), +, \times_n \subseteq V, \right.$$

$$A_{29} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & a_{29} & 0 \end{bmatrix} \middle| a_{29} \in [0, 12), +, \times_n \subseteq V \text{ and} \right.$$

$$A_{30} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 0 & a_{30} \end{bmatrix} \mid a_{30} \in [0, 12), +, \times_n \} \subseteq V \text{ are}$$

S-special interval pseudo linear subalgebras of  $V$  over the S-ring  $Z_{12}$ .

Clearly

$$A_i \cap A_j = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \text{ if } i \neq j, 1 \leq i, j \leq 30$$

and  $A_1 + A_2 + \dots + A_{30} = V$  is the direct sum of S-special interval pseudo linear subalgebras of  $V$ . Every  $A_i$  is infinite dimensional over  $R = Z_{12}$ .

We have 30 sublinear algebra of finite dimension but they will not lead to the direct sum.

$$\text{Let } B_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \mid a_1 \in Z_{12}, +, \times_n \} \subseteq V,$$

$$B_2 = \left\{ \begin{bmatrix} 0 & a_2 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \mid a_2 \in Z_{12}, +, \times_n \} \subseteq V,$$

$$B_3 = \left\{ \begin{bmatrix} 0 & 0 & a_3 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \mid a_3 \in Z_{12}, +, \times_n \} \subseteq V, \dots,$$



$$B_{28} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ a_{28} & 0 & 0 \end{bmatrix} \right\} \quad a_{28} \in Z_{12}, +, \times_n \subseteq V,$$

$$B_{29} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & a_{29} & 0 \end{bmatrix} \right\} \quad a_{29} \in Z_{12}, +, \times_n \subseteq V \text{ and}$$

$$B_{30} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 0 & a_{30} \end{bmatrix} \right\} \quad a_{30} \in Z_{12}, +, \times_n \subseteq V$$

be S-special interval pseudo linear subalgebras of  $V$ .

All of them are one dimensional over  $Z_{12}$ .

But  $B_1 + \dots + B_{30} \neq V$  however

$$B_i \cap B_j = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \quad \text{if } i \neq j, 1 \leq i, j \leq 30.$$

Likewise we can have infinite dimensional S-special and finite dimensional quasi vector spaces.

$$D_1 = \left\{ \begin{bmatrix} a_1 & a_2 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \right\} \quad a_1 \in [0, 12), a_2 \in \{0, 0.5, 1, 1.5, 2, \dots, \\ 11, 11.5\}, +, \times_n \subseteq V,$$

$$D_2 = \left\{ \begin{bmatrix} 0 & 0 & a_3 \\ a_4 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \middle| a_3 \in [0, 12), a_4 \in \{0, 0.2, 0.4, 0.6, 0.8, 1, \right.$$

$1.2, \dots, 11.2, 11.4, \dots, 11.8\} \subseteq [0, 12), +, \times_n\} \subseteq V$  and so on.

$$D_{14} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & a_{27} \\ a_{28} & 0 & 0 \end{bmatrix} \middle| a_{27} \in [0, 12), a_4 \in \{0, 0.1, 0.2, \dots, 1, 1.2, \right.$$

$\dots, 11, 11.1, \dots, 11.9\} \subseteq [0, 12), +, \times_n\} \subseteq V$  and

$$D_{15} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & a_{29} & a_{30} \end{bmatrix} \middle| a_{29} \in [0, 12) \text{ and } a_{30} \in \{0, 0.5, 1, 1.5, \dots, \right.$$

$11, 11.5\} \subseteq [0, 12), +, \times_n\} \subseteq V$

be the S-special interval quasi vector subspaces of  $V$  over  $Z_{12}$ .

None of them is a special interval pseudo linear subalgebra as product is not defined in  $D_i$  for the  $(D_i, \times_n)$  is not a semigroup that there exists  $A, B \in D_i$  such that  $A \times_n B \notin D_i$ , thus all  $D_i$ 's are only S-quasi vector subspaces and they are not pseudo linear subalgebras.

However

$$D_i \cap D_j = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \text{ for } i \neq j, 1 \leq i, j \leq 15.$$

Further  $D_1 + \dots + D_{15} \neq V$  so it is not a direct sum. All the 15 quasi vector subspaces are infinite dimensional over  $Z_{12}$ .

Let

$$W_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \middle| a_1 \in \{0, 0.5, 1, 1.5, \dots, 11, 11.5\} \subseteq \right.$$

$$[0, 12), +, \times_n \} \subseteq V,$$

$$W_2 = \left\{ \begin{bmatrix} 0 & a_2 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \middle| a_2 \in \{0, 0.5, 1, 1.5, \dots, 11, 11.5\} \subseteq \right.$$

$$[0, 12), +, \times_n \} \subseteq V,$$

$$W_3 = \left\{ \begin{bmatrix} 0 & 0 & a_3 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \middle| a_3 \in \{0, 0.5, 1, 1.5, \dots, 11, 11.5\} \subseteq [0, 12), \right.$$

$$+, \times_n \} \subseteq V \text{ and so on.}$$

$$W_{30} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 0 & a_{30} \end{bmatrix} \middle| a_{30} \in \{0, 0.5, 1, 1.5, \dots, 11, 11.5\} \subseteq [0,$$

$$12), +, \times_n \} \subseteq V$$

be S-special interval linear quasi vector subspace of  $V$  over  $R = Z_{12}$ . Clearly each  $W_i$  is finite dimensional over  $R = Z_2$ ;

$$W_i \cap W_j = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \text{ if } i \neq j, 1 \leq i, j \leq 30.$$

Further  $W = W_1 + \dots + W_{30} \subseteq V$  so is not a direct sum we see

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix} \right\} a_i \in \{0, 0.5, 1, 1.5, \dots, 11, 11.5\} \subseteq$$

$$[0, 12), +, \times_n, 1 \leq i \leq 30\} \subseteq V,$$

$W$  is a S-quasi subvector space of  $V$  and  $W$  is the subdirect subsum of  $W_i$ 's,  $1 \leq i \leq 30$ .

**Example 3.20:** Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & \dots & \dots & a_8 \\ a_9 & \dots & \dots & a_{12} \\ a_{13} & \dots & \dots & a_{16} \\ a_{17} & \dots & \dots & a_{20} \\ a_{21} & \dots & \dots & a_{24} \end{bmatrix} \right\} a_i \in [0, 15), 1 \leq i \leq 24, +, \times_n\}$$

be a S-special interval pseudo linear algebra over the S-ring  $Z_{15}$ .  $V$  has finite dimensional S-sublinear pseudo algebras and infinite dimensional S-linear pseudo subalgebras of  $V$ .

Infact  $V$  has finite dimensional  $S$ -quasi vector subspaces as well as infinite dimensional  $S$ -quasi vector subspaces.

Further  $V$  can be represented as a direct sum of sublinear algebras.

Let

$$M_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid a_i \in [0, 15), 1 \leq i \leq 4, +, \times_n \} \subseteq V$$

be a special interval linear pseudo subalgebra of infinite dimension over  $R = Z_{15}$ .

$$M_2 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid a_i \in [0, 15), 1 \leq i \leq 4, +, \times_n \} \subseteq V$$

is again a  $S$ -special interval pseudo linear subalgebra of  $V$  of infinite dimension over  $R = Z_{15}$ .

Let

$$M_3 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid a_i \in [0, 15), 1 \leq i \leq 4, +, \times_n \} \subseteq V$$

be a S-special interval pseudo linear subalgebra of  $V$  the S-ring  $R = Z_{15}$ .

$$M_4 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid a_i \in [0, 15), 1 \leq i \leq 4, +, \times_n \} \subseteq V$$

is again a S-special interval pseudo linear subalgebra of  $V$  over  $R = Z_{15}$ .

$$M_5 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid a_i \in [0, 15), 1 \leq i \leq 4, +, \times_n \} \subseteq V.$$

$$M_6 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \end{bmatrix} \mid a_i \in [0, 15), 1 \leq i \leq 4, +, \times_n \} \subseteq V$$

is again a S-special interval pseudo linear subalgebra of infinite dimension over  $R = Z_{15}$ .

All the six S-sublinear pseudo algebras are infinite dimension over  $R = Z_{15}$ .

$$\text{Further } M_i \cap M_j = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\} \text{ if } i \neq j, 1 \leq i, j \leq 6.$$

Also  $V = M_1 + M_2 + \dots + M_6$  is the direct sum of S-sublinear pseudo algebras.

However  $V$  can be represented as a direct sum in several ways.

We will now proceed onto describe S-special interval pseudo linear subalgebras of finite dimension over  $R = Z_{15}$ .

$$\text{Let } T_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid a_i \in Z_{15}, 1 \leq i \leq 4, +, \times_n \subseteq V, \right.$$

$$T_2 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid a_i \in Z_{15}, 1 \leq i \leq 4, +, \times_n \subseteq V, \right.$$

$$T_5 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid a_i \in Z_{15}, 1 \leq i \leq 4, +, \times_n \subseteq V \right.$$

and

$$T_6 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \end{bmatrix} \middle| a_i \in \mathbb{Z}_{15}, 1 \leq i \leq 4, +, \times_n \} \subseteq V$$

are the six S-special pseudo linear subalgebras of  $V$  over  $R = \mathbb{Z}_{15}$ .

We see all the S-sublinear pseudo algebras are of dimension four over  $R = \mathbb{Z}_{15}$ . But  $V \neq T_1 + \dots + T_6$  that is  $V$  cannot be written as a direct sum of  $T_i$ 's  $1 \leq i \leq 6$ .

Similarly  $V$  can have both finite and infinite dimensional S-special interval quasi vector subspaces of  $V$  which are not linear algebras of  $V$  over the S-ring  $\mathbb{Z}_{15}$ .

Let

$$B_1 = \left\{ \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| a_1 \in [0, 15), a_2 \in \{0, 0.5, 1, 1.5, 2, 2.5, \right.$$

$$\dots, 14, 14.5\} \subseteq [0, 15), +, \times_n \} \subseteq V$$

be a S-vector subspace and not a S-linear subalgebra of  $V$ .

For let

$$x = \begin{bmatrix} 9 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 3 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \in B_1.$$



We find

$$\begin{aligned}
 x \times_n y &= \begin{bmatrix} 9 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \times_n \begin{bmatrix} 3 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} 12 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \notin B_1,
 \end{aligned}$$

hence  $B_1$  is not a S-special interval pseudo linear subalgebra of  $V$  over  $R = Z_{15}$ . The dimension of  $V$  over  $R$  is infinite dimensional as a S-quasi vector subspace over  $R = Z_{15}$ .

Let

$$D_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid a_i \in \{0, 0.1, 0.2, \dots, 0.9, 1, 1.1, 1.2, \right.$$

$$\left. \dots, 1.9, 2, 2.1, \dots, 2.9, \dots, 14, 14.1, 14.2, \dots, 14.9\} \subseteq [0, 15), +, \times_n \subseteq V$$

be a S-special interval quasi vector subspace over  $R = Z_{15}$  it is not a S-linear subalgebra over the S-ring  $R = Z_{15}$ .

For if

$$x = \begin{bmatrix} 0.1 & 0.4 & 1.2 & 0.8 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 1.6 & 2.2 & 1.1 & 4.2 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \in D_1$$

we find

$$\begin{aligned} x \times_n y &= \begin{bmatrix} 0.1 & 0.4 & 1.2 & 0.8 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \times_n \begin{bmatrix} 1.6 & 2.2 & 1.1 & 4.2 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0.16 & 0.88 & 1.32 & 3.36 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \notin D_1 \end{aligned}$$

as the entries do not belong to  $\{0, 0.1, 0.2, 0.3, \dots, 1, 1.1, \dots, 14, 14.1, 14.2, \dots, 14.9\} \subseteq [0, 15)$ .

Hence  $D_1$  is not closed under product so is only a  $S$ -special quasi vector subspace of  $V$  over  $R = Z_{15}$ . However  $D_1$  is a finite dimensional  $S$ -special interval vector subspace of  $V$  over the  $S$ -ring  $R$ .

We have studied the notion of  $S$ -sublinear algebras of finite and infinite dimension and  $S$ -special pseudo linear subalgebras of finite dimension of  $V$  over the  $S$ -ring.

We can always write  $V$  as a direct sum of  $S$ -sublinear algebras over  $Z_{15}$ .

Further  $V$  has  $S$ -special interval quasi vector subspaces of both finite and infinite dimension over the  $S$ -ring  $Z_n$ .

Now we proceed onto study the  $S$ -special linear transformation and  $S$ -special linear operator of a  $S$ -special interval linear algebra (vector space) over a  $S$ -ring  $Z_n$ . We will also define  $S$ -special linear functional and so on. We can as in case of usual vector spaces define linear transformation in case of  $S$ -special interval vector spaces only if they are defined on the same  $S$ -ring  $Z_n$ .

Here we are going to define also the concept of special quasi induced linear transformation.

All these will be illustrated by some examples.

**Example 3.21:** Let  $V = \{(a_1, a_2, a_3) \mid a_i \in [0, 21), 1 \leq i \leq 3, +\}$  be a  $S$ -special interval vector space over the  $S$ -ring  $Z_{21}$ .

$$W = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_6 \end{bmatrix} \mid a_i \in [0, 21), 1 \leq i \leq 6, + \right\}$$

be a  $S$ -special interval vector space over the  $S$ -ring  $R = Z_{21}$ .

Define  $T_1 : V \rightarrow W$  by

$$T_1\{(a_1, a_2, a_3)\} = \begin{bmatrix} a_1 \\ 0 \\ a_2 \\ 0 \\ a_3 \\ 0 \end{bmatrix} \text{ for every } (a_1, a_2, a_3) \in V.$$

Clearly  $T_1$  is a S-special linear transformation of  $V$  to  $W$ .

We see  $\ker T_1 = \{(0, 0, 0)\}$ . In this way we can define several such S-special linear transformation from  $V$  to  $W$ .

We can have another S-special linear transformation  $T_2$  from  $V$  to  $W$  as follows.

$T_2 : V \rightarrow W$  by

$$T_2 \{(a_1, a_2, a_3)\} = \begin{bmatrix} a_1 + a_2 \\ a_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Clearly  $T_2$  is also a S-special linear transformation from  $V$  to  $W$  and  $\ker T_2 \neq \{(0, 0, 0)\}$ .

$$\text{For } \ker T_2 = \{x = (a_1, a_2, a_3) \in V \mid T_2(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\}.$$

$$\text{We see } T \{(a_1, a_2, a_3)\} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ if } a_1 + a_2 \equiv 0 \pmod{21} \text{ and } a_3 = 0.$$

Thus  $\ker T \neq \{(0, 0, 0)\}$  and  
 $\ker T = \{(a_1, a_2, a_3) \mid a_1 + a_2 = 0 \pmod{21} \text{ and } a_3 = 0\}$ .

Interested reader can define may transformation of this form.

We can define the S-special linear transformation from W to V as follows.

$$S_1 \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \right\} = (a_1 + a_2, a_3 + a_4, a_5 + a_6).$$

$S_1$  is a S-special linear transformation from W to V.

Further

$$\ker S_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \mid a_1 + a_2 = 0, a_3 + a_4 = 0, a_5 + a_6 = 0, a_i \in [0, 21), \right.$$

$$\left. 1 \leq i \leq 6 \right\} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ as } a_1 = 20 \text{ and } a_2 = 1, a_1 = 5 \text{ and } a_2 = 16$$

and so on like  $a_1 = 0.0003$  and  $a_2 = 20.9997$ ,  $a_3 + a_4 = 0$  where  $a_3 = 19.2$  and  $a_4 = 1.8$   $a_3 = 10$  and  $a_4 = 11$  and so on,  $a_5 = 0.04$  and  $a_6 = 20.46$  and so on.

Thus  $\ker S_1$  is non trivial;  $\ker S_1$  is a  $S$ -subspace of  $V$ .

For we have to show

- (1) if  $x, y \in \ker S_1$ ,  $x + y \in \ker S_1$
- (2) if  $x \in \ker S_1$  then  $-x \in \ker S_1$
- (3) if  $c \in Z_{21}$  and  $x \in \ker S_1$ ,  $cx \in \ker S_1$ .

All the three conditions can be easily proved without any difficulty. Hence the claim.

Suppose we define  $S_2 : W \rightarrow V$  by

$$S_2 \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \right\} = (a_1, a_2, a_3)$$

be a  $S$ -special linear transformation of  $W$  to  $V$ .

$$\text{Then also } \ker S_2 \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\ker S_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \mid a_i \in [0, 21); 4 \leq i \leq 6 \right\} \subseteq W$$

is a proper subspace of  $W$  over  $R = \mathbb{Z}_{21}$ .

Now  $S_1$  and  $S_2$  are  $S$ -special linear transformations from  $W$  to  $V$ .

**Example 3.22:** Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix} \mid a_i \in [0, 93), 1 \leq i \leq 16 \right\}$$

be the  $S$ -special subset interval vector space over the  $S$ -ring  $R = \mathbb{Z}_{93}$ .

$V$  can be written as a direct sum of  $S$ -subvector spaces over  $R = \mathbb{Z}_{93}$ .

$V$  has both  $S$ -subspaces of finite and infinite dimension over  $R = \mathbb{Z}_{93}$ ; on  $V$  we can define linear operator.

The linear operator in this case also is the same as the usual spaces.

Let

$$T_1 : V \rightarrow V \text{ defined by } T_1 \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix} \right\} = \begin{bmatrix} a_1 & 0 \\ 0 & a_4 \\ a_5 & 0 \\ 0 & a_6 \\ a_7 & 0 \\ 0 & a_8 \\ a_9 & 0 \\ 0 & a_{10} \end{bmatrix}.$$

It is easily verified  $T_1$  is a linear operator on  $V$  and

$$\ker T_1 \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}.$$

Now define  $T_2 : V \rightarrow V$

$$T_2 \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix} \right\} = \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \\ a_3 & a_4 \\ 0 & 0 \\ a_5 & a_6 \\ 0 & 0 \\ a_7 & a_8 \\ 0 & 0 \end{bmatrix}.$$



$$T_2 \text{ is also a linear operator on } V. \text{ ker } T_2 \neq \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \right\}$$

$$\text{Consider } T_1 \circ T_2, T_1 \circ T_2 \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix} \right\}.$$

$$T_2 \left( T_1 \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix} \right\} \right)$$

$$= T_2 \left( \begin{bmatrix} a_1 & 0 \\ 0 & a_4 \\ a_5 & 0 \\ 0 & a_6 \\ a_7 & 0 \\ 0 & a_8 \\ a_9 & 0 \\ 0 & a_{10} \end{bmatrix} \right) = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \\ a_5 & 0 \\ 0 & 0 \\ a_7 & 0 \\ 0 & 0 \\ a_9 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\text{Clearly } (T_1 \circ T_2) \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix} \right\} = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \\ a_5 & 0 \\ 0 & 0 \\ a_7 & 0 \\ 0 & 0 \\ a_9 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\text{Consider } (T_2 \circ T_1) \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix} \right\} =$$

$$T_1 \left\{ T_2 \left( \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix} \right) \right\} = T_1 \left\{ \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \\ a_3 & a_4 \\ 0 & 0 \\ a_5 & a_6 \\ 0 & 0 \\ a_7 & a_8 \\ 0 & 0 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \\ a_3 & a_4 \\ 0 & 0 \\ a_5 & a_6 \\ 0 & 0 \\ a_7 & a_8 \\ 0 & 0 \end{bmatrix}.$$

We see in this case  $T_1 \circ T_2 = T_2 \circ T_1$  and  $T_1 \circ T_2 : V \rightarrow V$  is again a linear operator on  $V$ .

Let  $T_3 : V \rightarrow V$  defined by

$$T_3 \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix} \right\} = \begin{bmatrix} a_1 & 0 \\ a_3 & 0 \\ \vdots & \vdots \\ a_{15} & 0 \end{bmatrix}; T_3 \text{ is a linear operator on } V.$$

$$T_3 \circ T_1 \left( \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix} \right) = T_1 \left( T_3 \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix} \right)$$

$$= T_1 \left\{ \begin{bmatrix} a_1 & 0 \\ a_3 & 0 \\ \vdots & \vdots \\ a_{15} & 0 \end{bmatrix} \right\} = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \\ a_5 & 0 \\ 0 & 0 \\ a_7 & 0 \\ 0 & 0 \\ a_9 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\text{Consider } T_1 \circ T_3 \left( \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix} \right) = T_3 \left( T_1 \left( \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix} \right) \right)$$

$$= T_3 \left( \begin{bmatrix} a_1 & 0 \\ 0 & a_3 \\ a_5 & 0 \\ 0 & a_8 \\ a_9 & 0 \\ 0 & a_{10} \\ a_{11} & 0 \\ 0 & a_{12} \end{bmatrix} \right) = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \\ a_5 & 0 \\ 0 & 0 \\ a_9 & 0 \\ 0 & 0 \\ a_{11} & 0 \\ 0 & 0 \end{bmatrix}.$$

In this case also we see  $T_1 \circ T_3 = T_3 \circ T_3$  and  $T_1 \circ T_3$  is a linear operator on  $V$ .

Now let

$$W_1 = \left\{ \begin{bmatrix} a_1 & 0 \\ a_2 & 0 \\ \vdots & \vdots \\ a_8 & 0 \end{bmatrix} \mid a_i \in [0, 93), 1 \leq i \leq 8 \right\} \subseteq V$$

be a  $S$ -special interval vector subspace of  $V$  over the  $S$ -ring.

We define  $T : V \rightarrow V$  by

$$T \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix} \right\} = \begin{bmatrix} a_1 & 0 \\ a_3 & 0 \\ \vdots & \vdots \\ a_{15} & 0 \end{bmatrix};$$

clearly  $T$  is a linear operator which is a projection of  $V$  onto the subspace  $W_1$ .

$$\begin{aligned} \text{Now } T \circ T \left( \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix} \right) &= T \left( \begin{bmatrix} a_1 & 0 \\ a_3 & 0 \\ \vdots & \vdots \\ a_{15} & 0 \end{bmatrix} \right). \\ &= \begin{bmatrix} a_1 & 0 \\ a_3 & 0 \\ \vdots & \vdots \\ a_{15} & 0 \end{bmatrix}. \end{aligned}$$

Thus  $T \circ T = T$  for  $(T \circ T) [(x)] = T(x)$  for all  $x \in V$ .

We can have the notion of projection of  $V$  onto a subspace of  $V$ .

**Example 3.23:** Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \mid a_i \in [0, 94), 1 \leq i \leq 12 \right\}$$

be a  $S$ -special interval vector space over the  $S$ -ring  $Z_{94}$ .

We see

$$\begin{aligned} P_1 &= \left\{ \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid a_1 \in [0, 94) \right\} \subseteq V, \\ P_2 &= \left\{ \begin{bmatrix} 0 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid a_1 \in [0, 94) \right\} \subseteq V, \dots, \end{aligned}$$

$$P_{15} = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{15} & 0 \end{bmatrix} \mid a_{15} \in [0, 94) \right\} \subseteq V \text{ and}$$

$$P_{16} = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{16} \end{bmatrix} \mid a_{16} \in [0, 94) \right\} \subseteq V$$

be the 16 S-subspaces of  $V$  over the same S-ring  $Z_{16}$ .

We see if

$$\begin{aligned} T_1 : V \rightarrow V \text{ such that } T_1 \left( \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \right) \\ = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

is a special linear operator on  $V$  and it is a projection to the space  $P_1$ .

Likewise  $T_{10} : V \rightarrow V$  given by

$$T_{10} \left( \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a_{10} & 0 & 0 \end{bmatrix};$$

$T_{10}$  is a projection to the space  $P_{10}$ .

Thus we have  $T_1, T_2, \dots, T_{12}$  to be S-special linear operators which are all projections of  $V$  to the subspaces of  $P_i$ . Further  $T : V \rightarrow V$  given by

$$T \left( \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \right) = \begin{bmatrix} a_1 & 0 & a_3 & 0 \\ 0 & a_6 & 0 & a_8 \\ a_9 & 0 & a_{11} & 0 \end{bmatrix}$$

is a linear operator on  $V$ ; we see

$$T \circ T_2 \left( \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \right) = T_2 \left( T \left( \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \right) \right)$$

$$= T_2 \left( \begin{bmatrix} a_1 & 0 & a_3 & 0 \\ 0 & a_6 & 0 & a_8 \\ a_9 & 0 & a_{11} & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$T_2 \circ T \left\{ \left( \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \right) \right\}$$

$$= T \left( T_2 \left( \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \right) \right)$$

$$= T \left( \begin{bmatrix} 0 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus  $T \circ T_2 = T_2 \circ T = T_0$ ; the zero  $S$ -linear transformation of  $V$  to  $V$ .

We see  $T_i \circ T_j = T_0$  if  $i \neq j$ , and  $T_i \circ T_i = T_i$ ;  $1 \leq i, j \leq 12$ .

This is the way we get projections depending on the subspaces.

All projections may not satisfy  $T_i \circ T_j = T_o$  ( $i \neq j$ ). This will not be true in the case of all S-subspaces of  $V$ .

For take

$$B_1 = \left\{ \begin{bmatrix} a_1 & a_2 & 0 & a_3 \\ 0 & 0 & a_4 & 0 \\ 0 & 0 & a_5 & a_6 \end{bmatrix} \mid a_i \in [0, 94), 1 \leq i \leq 6 \right\} \subseteq V$$

is a S-subspace of  $B$ .

$$B_2 = \left\{ \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & a_4 \end{bmatrix} \mid a_i \in [0, 94), 1 \leq i \leq 4 \right\} \subseteq V$$

be the S-subspace of  $V$  over the S-ring  $R = Z_{94}$ .

$S_1 : V \rightarrow V$  and

$$S_1 \left( \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \right) = \begin{bmatrix} a_1 & a_2 & 0 & a_3 \\ 0 & 0 & a_4 & 0 \\ 0 & 0 & a_5 & a_6 \end{bmatrix}$$

is a S-special linear operator which is also a projection of  $V$  to  $B_1$ .

Now consider

$$S_2 : V \rightarrow V \text{ be defined by } S_2 \left( \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \right)$$

$$= \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & a_4 \end{bmatrix};$$



$S_2$  is a S-special linear operator on  $V$  and is also a projection of  $V$  to  $B_2$ .

Now we see  $S_1 \circ S_2 \neq S_0 = T_0$ ; the zero S-linear operator on  $V$ .

$$\begin{aligned}
 \text{Consider } S_1 \circ S_2 & \left( \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \right) \\
 &= S_2 \{ S_4 \left( \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \right) \} \\
 &= S_2 \left( \begin{bmatrix} a_1 & a_2 & 0 & a_3 \\ 0 & 0 & a_4 & 0 \\ 0 & 0 & a_5 & a_6 \end{bmatrix} \right) \\
 &= \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & a_4 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
 \end{aligned}$$

$$\begin{aligned}
 \text{Consider } S_2 \circ S_1 & \left( \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \right) \\
 &= S_1 \{ S_2 \left( \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \right) \}
 \end{aligned}$$

$$= S_1 \left( \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & a_4 \end{bmatrix} \right) = \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & a_4 \end{bmatrix}.$$

Let  $W_1 : V \rightarrow V$  be defined by

$$W_1 \left( \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \right) = \begin{bmatrix} a_1 & 0 & a_3 & 0 \\ a_5 & 0 & a_7 & 0 \\ a_9 & 0 & a_{11} & 0 \end{bmatrix};$$

$W_1$  is a S-special linear operator on  $V$ .

Several types of linear operators can be defined on  $V$ .

For instance  $U : V \rightarrow V$  defined by

$$U \left( \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \right) = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix}$$

is a S-special linear operator on  $V$ .

$$\text{Clearly kernel } U \neq \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\}.$$

Thus we can have several types of linear operators.

**Example 3.24:** Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{bmatrix} \mid a_i \in [0, 58), 1 \leq i \leq 25 \right\}$$

be the S-special interval vector space over the S-ring  $R = Z_{58}$ .

Let  $T : V \rightarrow V$  be defined by

$$T\left(\begin{bmatrix} a_1 & \dots & a_5 \\ a_6 & \dots & a_{10} \\ \vdots & & \vdots \\ a_{21} & \dots & a_{25} \end{bmatrix}\right) = \begin{bmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_5 & 0 \\ 0 & 0 & 0 & 0 & a_6 \end{bmatrix}$$

is a S-special linear operator on  $V$ .

Define  $T_1 : V \rightarrow V$  be

$$T_1\left(\begin{bmatrix} a_1 & \dots & a_5 \\ a_6 & \dots & a_{10} \\ \vdots & & \vdots \\ a_{21} & \dots & a_{25} \end{bmatrix}\right) = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

$T_1$  is S-special linear operator on  $V$ .

$$\text{Clearly } \ker T_1 \neq \begin{bmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix}.$$

Define  $T_2 : V \rightarrow V$  is defined by

$$T_2 \left( \begin{bmatrix} a_1 & a_2 & \dots & a_5 \\ a_6 & a_7 & \dots & a_{10} \\ \vdots & \vdots & & \vdots \\ a_{21} & a_{22} & \dots & a_{25} \end{bmatrix} \right) = \begin{bmatrix} a_1 & 0 & 0 & 0 & 0 \\ a_2 & a_3 & 0 & 0 & 0 \\ a_4 & a_5 & a_6 & 0 & 0 \\ a_7 & a_8 & a_9 & a_{10} & 0 \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{bmatrix};$$

$T$  is a S-special linear operator on  $V$ .

$$\ker T_2 \neq \begin{bmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix}.$$

$T_3 : V \rightarrow V$  is defined by

$$T_3 \left( \begin{bmatrix} a_1 & a_2 & \dots & a_5 \\ a_6 & a_7 & \dots & a_{10} \\ \vdots & \vdots & & \vdots \\ a_{21} & a_{22} & \dots & a_{25} \end{bmatrix} \right) = \begin{bmatrix} 0 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & a_5 & a_6 & a_7 \\ 0 & 0 & 0 & a_8 & a_9 \\ 0 & 0 & 0 & 0 & a_{10} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is again a S-special linear operator on  $V$  and

$$\ker T_3 \neq \begin{bmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix}.$$

Next we proceed onto derive other properties associated with these S-special type of vector spaces.

We can also define the notion of S-special interval pseudo linear algebra over  $Z_n$  ( $n < \infty$ ) and  $Z_n$  a S-ring.

In the first place S-special interval vector space  $V$  over a S-ring;  $R = Z_n$  is defined as the S-special interval pseudo linear algebra if  $V$  is endowed with a product  $\cdot$  such that for  $a, b \in V$ .  $a \cdot b \in V$  and  $\cdot$  is associative on  $V$  and for  $a \in R$ ,  $v_1, v_2 \in V$ .

$a(v_1 + v_2) \neq av_1 + av_2 \in V$  in general the distributive law may or may not be true.

We will illustrate this situation by some examples.

**Example 3.25:** Let  $V = \{[0, 22), +, \times\}$  be the S-special pseudo interval linear algebra over the S-ring  $R = Z_{22}$ .

If  $x = 11.5$  and  $y = 5 \in V$ .  $x \cdot y = 11.5 \times 5 = 57.5 = 13.5 \in V$ .

This is the way  $V$  is a S-special pseudo interval linear algebra over the S-ring  $Z_{22}$ .

We see  $V$  is an infinite dimensional over the S-ring  $R = Z_{22}$ . We see  $V$  has S-special interval pseudo linear subalgebras over  $R = Z_{22}$  of finite order over  $R$ .

$M = \{Z_{22}, +, \times\}$ , the S-special interval pseudo linear subalgebra over  $R$  is of dimension one over  $R = Z_{22}$ .

$P = \{[0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, \dots, 20, 20.5, 21, 21.5]\} \subseteq V$  is a S-special interval pseudo linear subalgebra of finite dimension over  $R = Z_{22}$ .

**Example 3.26:** Let  $V = \{[0, 12), +, \times\}$  be the S-special pseudo interval linear algebra over the S-ring  $R = Z_{12}$ .  $V$  has finite dimensional sublinear pseudo algebras over the S-ring  $R = Z_{12}$ .

$M_1 = \{Z_{12}, \times, +\} \subseteq V$  is a S-special interval pseudo sublinear algebra of  $V$  over the S-ring of dimension 1 over  $R = Z_{12}$ .

$M_2 = \{\{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, \dots, 11, 11.2, 11.4, 11.6, 11.8\} \subseteq [0, 12)\} \subseteq V$  is a S-special quasi vector subspace of finite dimension over the S-ring  $R = Z_{12}$ .

Let  $W = \{\{0, 0.1, 0.2, \dots, 0.9, 1, 1.1, 1.2, \dots, 11.9\} \subseteq [0, 12)\} \subseteq V$  be a finite dimensional quasi vector subspace of  $V$  over  $R = Z_{12}$ .

However  $T_1 = \{\{0, 4, 8\} \subseteq Z_{12} \subseteq [0, 12)\}$  is a finite dimensional S-linear subalgebra of  $V$  over  $Z_{12}$ .

Likewise  $T_2 = \{\{0, 2, 4, 6, 8, 10\} \subseteq Z_{12} \subseteq [0, 12)\} \subseteq V$  is again a S-special pseudo sublinear algebra of  $V$  over  $R = Z_{12}$ .

$T_3 = \{\{0, 3, 6, 9\} \subseteq Z_{12} \subseteq [0, 12)\} \subseteq V$  is also a finite dimensional S-special pseudo linear subalgebra of  $V$  over  $R = Z_{12}$ .

$V$  has finite dimensional S-special linear pseudo subalgebras as well as S-special quasi vector subspaces over the S-ring  $Z_{12}$ .

**Example 3.27:** Let  $V = \{[0, 46), +, \times\}$  be the S-special pseudo interval linear algebra over the S-ring  $Z_{46}$ .

$V$  has finite dimensional S-special linear subalgebras of finite order though  $V$  is an S-infinite dimensional linear algebra over the S-ring  $Z_{46}$ .

Further  $P_1 = \{Z_{46}, +, \times\} \subseteq V$  is a one dimensional S-special linear subalgebra of dimension one over the S-ring  $Z_{46}$ .

We see  $M_1 = \{\{0, 0.5, 1, 1.5, 2, \dots, 44, 44.5, 45, 45.55\} \subseteq [0, 46)\} \subseteq V$  is not a S-special linear pseudo subalgebra only a S-special quasi vector subspace of  $V$  over  $R = Z_{46}$ .

However  $M_1$  is a finite dimensional S-quasi vector subspace of  $V$  over the S-ring  $R = Z_{46}$ .

$M_2 = \{\{0, 0.1, 0.2, 0.3, \dots, 0.9, 1, 1.1, 1.2, \dots, 44, 44.1, \dots, 44.9, 45, 45.1, \dots, 45.9\} \subseteq [0, 46)\} \subseteq V$  is a S-quasi vector subspace of  $V$  over the S-ring  $R = Z_{46}$ .

$M_2$  is a finite dimensional S-quasi vector subspace over the S-ring.  $M_3 = \{\{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, \dots, 1.8, 2, 2.2, \dots, 45, 45.2, \dots, 45.4, 45.6, 45.8\} \subseteq [0, 46)\}$  is a S-special quasi vector subspace of  $V$  over the S-ring  $Z_{46}$ .

Clearly  $M_3$  is also finite dimensional over  $R = Z_{46}$ .  $T_1 = \{\{0, 23\} \subseteq Z_{46} \subseteq [0, 46)\} \subseteq V$  is again a S-special sublinear algebra of  $V$  over the S-ring  $Z_{46}$  and

$T_2 = \{\{0, 2, 4, 6, \dots, 44\} \subseteq [0, 46)\} \subseteq V$  is again a S-special sublinear algebra of  $V$  over the S-ring  $Z_{46}$ , both  $T_1$  and  $T_2$  are both finite dimensional S-special pseudo linear subalgebra of  $V$  over the S-ring  $R = Z_{46}$ .

**Example 3.28:** Let  $V = \{[0, 93), ., +, \times\}$  be the S-special pseudo interval linear algebra over the S-ring  $R = Z_{93}$ .

This S-special interval pseudo linear algebras has both S-linear subalgebras as well as S-quasi vector subspaces which are finite dimensional over  $R$ .

Now we proceed onto give S-special pseudo linear algebras built using the interval  $[0, n)$  where  $Z_n$  is a S-ring.

**Example 3.29 :** Let  $V = \{(a_1, a_2, a_3, a_4) \mid a_i \in [0, 42), 1 \leq i \leq 4\}$  be a S-special pseudo linear algebra over the S-ring  $R = Z_{42}$ .  $V$  has both finite and infinite dimensional linear subalgebras over  $R = Z_{42}$ .

Further  $V$  can be written as a direct sum of S-sublinear algebras.

Let  $W_1 = \{(a_1, 0, 0, 0) \mid a_1 \in [0, 42), +, \times\} \subseteq V$  be a S-special interval pseudo sublinear algebra of infinite dimension over  $R = Z_{42}$ .

Let  $W_2 = \{(0, a_2, 0, 0) \mid a_2 \in [0, 42), +, \times\} \subseteq V$ ,  
 $W_3 = \{(0, 0, a_3, 0) \mid a_3 \in [0, 42), +, \times\} \subseteq V$  and  
 $W_4 = \{(0, 0, 0, a_4) \mid a_4 \in [0, 42), +, \times\} \subseteq V$  be the four S-special interval pseudo linear subalgebras of  $V$  over  $R = Z_{42}$ .

Clearly  $W_i \cap W_j = (0, 0, 0, 0)$  if  $i \neq j$ ,  $1 \leq i, j \leq 4$  and  $V = W_1 + W_2 + W_3 + W_4$  is a direct sum of sublinear pseudo algebras over the S-ring  $R = Z_{42}$ .

Let  $P_1 = \{(a_1, 0, 0, 0) \mid a_1 \in Z_{42}, +, \times\} \subseteq V$ ,  
 $P_2 = \{(0, a_2, 0, 0) \mid a_2 \in Z_{42}, +, \times\} \subseteq V$ ,  
 $P_3 = \{(0, 0, a_3, 0) \mid a_3 \in Z_{42}, +, \times\} \subseteq V$  and  
 $P_4 = \{(0, 0, 0, a_4) \mid a_4 \in Z_{42}, +, \times\} \subseteq V$  are the four S-special interval pseudo linear subalgebras of  $V$  over  $R = Z_{42}$ .

Clearly  $P_i \cap P_j = \{(0, 0, 0, 0)\}$  if  $i \neq j$ ,  $1 \leq i, j \leq 4$  and  $P = P_1 + P_2 + P_3 + P_4 \subsetneq V$  is also a finite dimensional linear subalgebra of  $V$  over  $R = Z_{42}$ . Thus this sort of direct sum we define as sub subdirect sum of sublinear algebras of  $V$ .

Let  $M_1 = \{(a_1, a_2, 0, 0) \mid a_1, a_2 \in \{0, 0.5, 1, 1.5, 2, 2.5, \dots, 41, 41.5\}, +, \times\} \subseteq V$  be a S-quasi special vector subspace of  $V$  over the S-ring  $Z_{42}$ .

$M_1$  is finite dimensional;  $M_1$  is not a linear pseudo subalgebra for if  $x = (0.5, 2.5, 0, 0)$  and  $y = (1.5, 2.5, 0, 0) \in M_1$ , then  $x \times y = (0.5, 2.5, 0, 0) \times (1.5, 2.5, 0, 0) = (7.25, 6.25, 0, 0) \notin M_1$ .

Hence the claim.

Let  $N_1 = \{(a_1, 0, a_2, 0) \mid a_1 \in [0, 42) \ a_2 \in \{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, \dots, 40, 40.2, \dots, 40.8, 41, 41.2, 41.4, 41.6, 41.8\} \subseteq$



$[0, 42), +, \times] \subseteq V$  be the only S-special interval quasi vector subspace of  $V$  over  $R = Z_{42}$ .

Infact  $N_1$  is an infinite dimensional S-special quasi interval vector subspace of  $V$  over  $R = Z_{42}$ .

Clearly  $N_1$  is a S-special interval pseudo linear subalgebra of  $V$  over the S-ring  $Z_{42}$ .

$$\begin{aligned} \text{Let } x &= (5, 0, 0.4, 0) \text{ and } y = (3, 0, 2.4, 0) \in N_1; \\ x \times y &= (5, 0, 0.4, 0) \times (3, 0, 2.4, 0) \\ &= (15, 0, 0.96, 0) \notin N_1. \end{aligned}$$

Hence the claim.

Thus we have S-special quasi subset vector subspaces of  $V$  of both finite and infinite dimension over the S-ring  $R = Z_{42}$ .

Let  $S_1 = \{(a_1, a_2, 0, 0) \mid a_1, a_2 \in Z_{42}\} \subseteq V$  be a S-special interval sublinear algebra of finite dimension over  $R = Z_{42}$ .  
 $S_2 = \{(a_1, a_2, 0, 0) \mid a_1, a_2 \in \{0, 0.5, 1, 1.5, 2, 2.5, \dots, 41, 41.5\} \subseteq [0, 42)\} \subseteq V$  is only a S-special quasi vector subspace of  $V$  over the S-ring  $R = Z_{42}$ .

Clearly  $S_2$  is not a S-special linear subalgebra of  $V$  for if  $x = (0.5, 1.5, 0, 0)$  and  $y = (0.5, 0.5, 0, 0) \in S_2$  then  
 $x + y = (0.5, 1.5, 0, 0) + (0.5, 0.5, 0, 0) = (1, 2, 0, 0) \in S_2$  but  
 $x \times y = (0.5, 1.5, 0, 0) \times (0.5, 0.5, 0, 0) = (0.25, 0.75, 0, 0) \notin S_2$ .

So  $S_2$  is not closed under the product hence  $S_2$  is only a S-special quasi vector subspace of  $V$  over  $R = Z_{42}$ .

However dimension of  $S_2$  over the ring  $R$  is finite.

Let  $S_3 = \{(0, a_1, 0, a_2) \mid a_1, a_2 \in \{0, 0.1, 0.2, \dots, 0.9, 1, 1.1, \dots, 1.9, \dots, 41.1, 41.2, \dots, 41.9\} \subseteq [0, 42)\} \subseteq V$  be the only finite dimensional S-special quasi vector subspace of  $V$  and is not a S-special pseudo linear subalgebra of  $V$  over the S-ring  $R = Z_{42}$ .

We have seen both finite and infinite dimensional S-special quasi vector subspaces of  $V$ . Now consider  $S_4 = \{(a_1, a_2, a_3, a_4) \mid a_i \in \{0, 2, 4, 6, 8, \dots, 40\} \subseteq Z_{42} \subseteq [0, 42); 1 \leq i \leq 4\} \subseteq V$  is a S-special linear subalgebra of  $V$  over the S-ring  $Z_{42}$ . Clearly dimension of  $S_4$  over  $Z_{42}$  is finite.

We have only finite number of S-special linear subalgebras of  $V$  over the S-ring  $R = Z_{42}$ .

**Example 3.30:** Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_6 \end{bmatrix} \mid a_i \in [0, 6); 1 \leq i \leq 6 \right\}$$

be the S-special vector space over the S-ring  $R = Z_6$ .

$V$  is infinite dimensional over the S-ring  $R = Z_6$ .  $V$  has several infinite dimensional S-special vector subspaces over  $Z_6$ .

For

$$P_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \mid a_1 \in [0, 6) \right\} \subseteq V$$

is a S-special vector subspace of  $V$  over the S-ring  $R = Z_6$ .

$$P_2 = \left\{ \begin{bmatrix} p_1 \\ p_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \mid p_1, p_2 \in [0, 6) \right\} \subseteq V$$

is a S-special vector subspace of  $V$  over the S-ring  $R = Z_6$  of infinite dimension over  $Z_6$ .

$$P_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ a_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_1 \in [0, 6) \right\} \subseteq V$$

is a S-special vector subspace of  $V$  over the S-ring  $R = Z_6$ .

$$P_4 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \mid a_i \in [0, 6), 1 \leq i \leq 3 \right\} \subseteq V$$

is a S-special vector subspace of  $V$  over the S-ring  $R = Z_6$ .

$$P_5 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ a_2 \\ 0 \\ a_3 \\ 0 \end{bmatrix} \mid a_i \in [0, 6), 1 \leq i \leq 3 \right\} \subseteq V$$

is a S-special vector subspace of  $V$  over the S-ring  $R = Z_6$ .

Thus  $V$  has several S-special vector subspace of infinite dimension over the S-ring  $Z_6$ .

Consider

$$S_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \mid a_1 \in Z_6 \right\} \subseteq V$$

is a S-special vector subspace of  $V$  over the S-ring  $Z_6$  and is finite dimensional over  $Z_6$ .

Let

$$S_2 = \left\{ \begin{bmatrix} 0 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \mid a_2 \in Z_6 \right\} \subseteq V$$

is a S-special vector subspace of  $V$  over the S-ring  $Z_6$  and is finite dimensional over  $Z_6$ .

$$S_3 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ a_2 \\ 0 \\ a_3 \\ 0 \end{bmatrix} \mid a_i \in Z_6, 1 \leq i \leq 3 \right\} \subseteq V$$

is a S-special vector subspace of  $V$  over  $Z_6$  of finite dimension.

$$S_4 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \end{bmatrix} \mid a_1, a_2 \in \mathbb{Z}_6 \subseteq [0, 6) \right\} \subseteq V$$

is a S-special vector subspace of finite dimension over the S-ring  $\mathbb{Z}_6$ .

$$S_5 = \left\{ \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_1 \end{bmatrix} \mid a_1 \in \mathbb{Z}_6 \right\} \subseteq V$$

is a finite dimensional S-special vector subspace of  $V$  over the S-ring  $\mathbb{Z}_6$ .

We can make  $V$  into a S-special pseudo linear algebra by defining a product  $\times_n$ , the natural product on  $V$ .

Thus  $(V, \times_n)$  becomes a S-special interval pseudo linear algebra of infinite dimension over  $\mathbb{Z}_6$ .

However  $V$  has also finite dimensional S-special linear subalgebras over  $\mathbb{Z}_6$ .

Let

$$M_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \mid a_1 \in \mathbb{Z}_6, \times_n \right\} \subseteq V$$

be a S-special linear subalgebra of  $V$  over the S-ring  $Z_6$ .  $M_1$  is finite dimensional over  $Z_6$ .

$$M_2 = \left\{ \begin{bmatrix} 0 \\ a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_1, a_2 \in Z_6, \times_n \} \subseteq V$$

is a S-special linear subalgebra of  $V$  over the S-ring  $Z_6$ .

$M_2$  is of dimension two over  $Z_6$ .

$$\text{The basis of } M_2 \text{ is } \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \subseteq M_2 \text{ over } Z_6.$$

$$M_3 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_6 \end{bmatrix} \mid a_i \in Z_6, 1 \leq i \leq 6 \right\} \subseteq V$$

is a S-special pseudo linear subalgebra of  $V$  over the S-ring  $Z_6$ .

Dimension of  $M_3$  over the S-ring  $Z_6$  is 6. The basis of  $M_3$  is given by

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \subseteq M_3 \text{ over } Z_6.$$

$$M_4 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_6 \end{bmatrix} \mid a_i \in \{0, 2, 4\} \subseteq Z_6, 1 \leq i \leq 6 \right\} \subseteq V$$

is a S-special linear subalgebra of  $V$  over  $Z_6$ .

Clearly  $M_4$  is of dimension 6 over  $Z_6$ .

$$\text{A basis of } M_4 \text{ is } \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \right\} \subseteq V.$$

$$M_5 = M_4 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_i \in \{0, 3\}, 1 \leq i \leq 3 \right\} \subseteq V$$

is a S-special linear subalgebra of  $V$  over the S-ring  $Z_6$ .  $M_5$  is finite dimensional over  $Z_6$ .

$$\text{A basis of } M_5 \text{ over } V \text{ is } \left\{ \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Thus  $V$  has S-special sublinear algebras of finite dimension as well as infinite dimension over  $Z_6$ .

$$T_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ a_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_1, a_2 \in \{0, 0.5, 1, 1.5, 2, 2.5, \dots, 4.5, 5, \right.$$

$$5.5\} \subseteq [0, 6) \subseteq V$$

is only a S-special quasi vector subspace of  $V$  of finite dimension over  $Z_6$ .

Clearly  $T_2$  is not a S-special linear subalgebra of  $V$  over  $Z_6$ .

$$\text{For take } x = \begin{bmatrix} 0.5 \\ 0 \\ 1.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 1.5 \\ 0 \\ 2.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in T_1.$$



$$\text{We see } x \times_n y = \begin{bmatrix} 0.5 \\ 0 \\ 1.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} \times_n \begin{bmatrix} 1.5 \\ 0 \\ 2.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0 \\ 3.75 \\ 0 \\ 0 \\ 0 \end{bmatrix} \notin T_2;$$

hence  $T_2$  is not a  $S$ -special linear subalgebra only a  $S$ -special quasi vector subspace of  $V$  over  $R = Z_6$  is of finite dimension.

$$\text{Let } T_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \end{bmatrix} \mid a_1, a_2 \in \{0, 0.01, 0.02, \dots, 0.1, \dots, 1.0, 1.01, \dots, 5.01, 5.02, \dots, 5.99\} \subseteq [0, 6), + \right\} \subseteq V$$

be a  $S$ -special quasi vector subspace of  $V$  over the  $S$ -ring  $Z_6$ .

Clearly  $T_3$  is a  $S$ -special linear subalgebra of  $V$  over  $Z_6$ .

$$\text{Let } T_4 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_2 \end{bmatrix} \mid a_1, a_2 \in \{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, \dots, 5, 5.2, 5.4, 5.6, 5.8\} \subseteq [0, 6) \right\} \subseteq V$$

be a  $S$ -special quasi vector subspace of  $V$  over the  $S$ -ring  $Z_6$  and is not a  $S$ -special linear subalgerba of  $V$  over  $Z_6$ .

Clearly  $T_4$  is finite dimensional over  $Z_6$ .

$$\text{Let } T_5 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \quad a_1 \in \{0, 0.25, 0.50, 0.75, 1, 1.25, \dots,$$

$$5, 5.25, 5.5, 5.75\} \subseteq [0, 6),$$

and  $a_2 \in \{0, 0.001, 0.002, \dots, 1, 1.001, 1.002, \dots, 4.001, \dots, 4.999\} \subseteq [0, 6)\} \subseteq V$  be a S-special interval quasi vector subspace of  $V$  over the S-ring  $Z_6$ .

Clearly dimension of  $T_5$  over  $Z_6$  is finite dimensional.

We can write  $V = W_1 + \dots + W_6$  or

$V = W_1 + W_2$  or

$V = W_1 + W_2 + W_3$  or

$V = W_1 + W_2 + W_3 + W_4$  and

$V = W_1 + W_2 + W_3 + W_4 + W_5$ .

We just show this by some illustration.

$$\text{Let } W_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\} \quad a_1 \in [0, 6), \times_n \} \subseteq V,$$

$$W_2 = \left\{ \begin{bmatrix} 0 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \mid a_2 \in [0, 6), \times_n \} \subseteq V,$$

$$W_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ a_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_3 \in [0, 6), \times_n \} \subseteq V,$$

$$W_4 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_4 \\ 0 \\ 0 \end{bmatrix} \mid a_4 \in [0, 6), \times_n \} \subseteq V,$$

$$W_5 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_5 \\ 0 \end{bmatrix} \mid a_5 \in [0, 6), \times_n \} \subseteq V \text{ and}$$

$$W_6 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_6 \end{bmatrix} \mid a_6 \in [0, 6), \times_n \} \subseteq V$$

be the six S-special pseudo linear subalgebras of  $V$  over the S-ring  $Z_6$ .

Clearly

$$W_i \cap W_j = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ if } i \neq j, 1 \leq i, j \leq 6$$

and  $V = W_1 + \dots + W_6$  and thus  $V$  is the direct sum of S-special linear subalgebras over  $Z_6$ .

Let

$$B_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_1, a_2 \in [0, 6), \times_n \} \subseteq V,$$

$$B_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ a_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_3 \in [0, 6), \times_n \} \subseteq V,$$

$$B_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_4 \\ 0 \\ 0 \end{bmatrix} \mid a_4 \in [0, 6), \times_n \} \subseteq V,$$

$$B_4 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_5 \\ 0 \end{bmatrix} \mid a_5 \in [0, 6), \times_n \} \subseteq V,$$

$$B_5 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_6 \end{bmatrix} \mid a_6 \in [0, 6), \times_n \} \subseteq V$$

be the S-special linear subalgebras of  $V$  over the S-ring  $Z_6$ .

We see

$$B_i \cap B_j = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ if } i \neq j, 1 \leq i, j \leq 6 \text{ and}$$

$V = B_1 + B_2 + B_3 + B_4 + B_5$  is the direct sum of S-special sublinear algebras of  $V$ .

Let

$$C_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ a_2 \\ 0 \\ a_3 \\ 0 \end{bmatrix} \mid a_i \in [0, 6), 1 \leq i \leq 3, \times_n \subseteq V \text{ and} \right.$$

$$C_2 = \left\{ \begin{bmatrix} 0 \\ a_1 \\ 0 \\ a_2 \\ 0 \\ a_3 \end{bmatrix} \mid a_i \in [0, 6), 1 \leq i \leq 3, \times_n \subseteq V \right.$$

be the two S-special linear subalgebras of  $V$  over  $Z_6$ .

We have

$$C_1 \cap C_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ and } V = C_1 + C_2$$

is the direct sum of S-sublinear algebras of V.

$$\text{Let } D_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_2 \end{bmatrix} \mid a_1, a_2 \in [0, 6), \times_n \} \subseteq V,$$

$$D_2 = \left\{ \begin{bmatrix} 0 \\ a_1 \\ 0 \\ 0 \\ a_2 \\ 0 \end{bmatrix} \mid a_1, a_2 \in [0, 6), \times_n \} \subseteq V \text{ and}$$

$$D_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ a_1 \\ a_2 \\ 0 \\ 0 \end{bmatrix} \mid a_1, a_2 \in [0, 6), \times_n \} \subseteq V$$

be S-special linear subalgebras of V over the S-ring  $Z_6$ .

Clearly

$$D_i \cap D_j = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}, i \neq j, 1 \leq i, j \leq 3 \text{ and}$$

$D_1 + D_2 + D_3 = V$  is the direct sum of S-sublinear algebras.  
Let

$$E_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_1, a_2 \in Z_6 \right\} \subseteq V,$$

$$E_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \mid a_1, a_2, a_3 \in Z_6 \right\} \subseteq V \text{ and}$$

$$E_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_1 \end{bmatrix} \mid a_1 \in Z_6 \right\} \subseteq V$$



be the three S-special linear subalgebras of  $V$ .

$$E_i \cap E_j = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}, i \neq j, 1 \leq i, j \leq 3.$$

But  $V \neq E_1 + E_2 + E_3$ .

$$\text{Further } E = E_1 + E_2 + E_3 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \mid a_i \in \mathbb{Z}_6, 1 \leq i \leq 6 \right\} \subseteq V$$

is a S-special pseudo linear subalgebra of finite dimension over  $R = \mathbb{Z}_6$ .

Now we give some more examples before we proceed onto discuss about other properties.

**Example 3.31:** Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ \vdots & \vdots & \vdots & \vdots \\ a_{49} & a_{50} & a_{51} & a_{52} \end{bmatrix} \mid a_i \in [0, 46), 1 \leq i \leq 52 \right\}$$

be the S-special vector space over the S-ring  $R = \mathbb{Z}_{46}$ . If the natural product  $\times_n$  on matrices is defined  $V$  becomes a S-special pseudo linear algebra over the S-ring  $R = \mathbb{Z}_{46}$ .

It is easily verified  $V$  can be written as a direct sum.  $V$  has both finite and infinite dimensional  $S$ -special linear subalgebras. Also  $V$  has both finite and infinite dimensional  $S$ -special quasi vector subspaces over  $Z_{46}$ .

Let

$$W_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid a_i \in \{0, 0.5, 1, 1.5, 2, \dots, 40, \right.$$

$$40.5, 41, 41.5, 42, 42.5, \dots, 45, 45.5\} \mid 1 \leq i \leq 4 \} \subseteq V$$

be a  $S$ -special quasi vector subspace of  $V$  and is not a  $S$ -special linear subalgebra of  $V$ .

Clearly dimension of  $W_1 \subseteq V$  over  $R = Z_{46}$  is finite dimensional.

Let

$$W_2 = \left\{ \begin{bmatrix} a_1 & 0 & 0 & a_2 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid a_i \in \{0, 0.2, 0.4, 0.6, 0.8, 1, \dots, \right.$$

$$45, 45.2, \dots, 45.8\} \text{ and } a_2 \in [0, 46) \} \subseteq V.$$

$W_2$  is a  $S$ -special quasi vector subspace of  $V$  over  $R = Z_{46}$ . The dimension of  $W_2$  over  $Z_{46}$  is infinite.

However all the  $S$ -special interval pseudo linear algebras defined over the  $S$ -ring in this chapter are commutative, now we proceed onto give examples of non commutative  $S$ -special interval pseudo linear algebras over the  $S$ -ring.

**Example 3.32:** Let

$V = \{(a_1 \mid a_2 \ a_3 \ a_4 \mid a_5) \mid a_i \in [0, 58), 1 \leq i \leq 5, +, \times\}$  be the S-special interval pseudo linear algebra over the S-ring  $Z_{58}$ .

$V$  has both finite and infinite dimensional S-special linear subalgebras. Further  $V$  has both finite and infinite dimensional S-special quasi vector subspaces.

Take  $P_1 = \{(a_1 \mid a_2 \ a_3 \ a_4 \mid a_5) \mid a_i \in Z_{58}, 1 \leq i \leq 5\} \subseteq V$ ,  $P_1$  is a finite dimensional S-special linear subalgebra of  $V$  over the S-ring  $R = Z_{58}$ .

Let  $P_2 = \{(a_1 \mid a_2 \ 0 \ 0 \mid a_3) \mid a_i \in [0, 58), 1 \leq i \leq 3\} \subseteq V$ ;  $P_2$  be a infinite dimensional S-special linear subalgebra of  $V$  over the S-ring  $Z_{58}$ .

Let  $P_3 = \{(a_1 \mid 0 \ 0 \ a_2 \mid 0) \mid a_1, a_2 \in \{0, 0.5, 1, 1.5, 2, 2.5, \dots, 57, 57.5\} \subseteq [0, 58)\} \subseteq V$  be a S-special quasi vector subspace of  $V$  over S-ring  $Z_{58}$ .  $P_3$  is finite dimensional over S-ring  $Z_{58}$ .

$P_4 = \{(0 \mid a_1 \ a_2 \ a_3 \mid 0) \mid a_1, a_2 \in [0, 58), a_3 \in \{0, 0.1, 0.2, \dots, 0.9, 1, 1.1, \dots, 57.1, 57.2, \dots, 57.9\} \subseteq [0, 58)\} \subseteq V$  is an infinite dimensional S-special quasi vector subspace of  $V$  over  $Z_{58}$ .

**Example 3.33:** Let

$$V = \left\{ \left[ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{array} \right] \mid a_i \in [0, 51), 1 \leq i \leq 8, +, \times_n \right\}$$

be a S-special pseudo interval linear algebra over the S-ring  $R = Z_{51}$  under the natural product of matrices.

We have finite and infinite dimensional S-special pseudo linear subalgebras as well as finite and infinite dimensional S-special quasi vector subspaces.

$$\text{Let } T_1 = \left\{ \begin{bmatrix} 0 \\ \overline{a_1} \\ a_2 \\ \overline{0} \\ 0 \\ 0 \\ \overline{0} \\ 0 \end{bmatrix} \mid a_1, a_2 \in [0, 51), +, \times_n \} \subseteq V \text{ and}$$

$$T_2 = \left\{ \begin{bmatrix} 0 \\ \overline{0} \\ 0 \\ \overline{0} \\ 0 \\ 0 \\ \overline{a_1} \\ a_2 \end{bmatrix} \mid a_1, a_2 \in [0, 51), +, \times_n \} \subseteq V$$

be two S-special pseudo interval linear subalgebra of  $V$ .

We see both  $T_1$  and  $T_2$  are infinite dimensional S-special interval linear subalgebras of  $V$  over the S-ring  $Z_{51}$ .

It is observed that for every  $x \in T_1$  and every  $y \in T_2$  are such that

$$x \times_n y = \begin{bmatrix} 0 \\ \overline{0} \\ 0 \\ \overline{0} \\ 0 \\ 0 \\ \overline{0} \\ 0 \end{bmatrix}.$$

Thus the S-special linear subalgebra  $T_1$  is orthogonal to the S-special linear subalgebra  $T_2$ , however  $T_1$  is not the orthogonal complement of  $T_2$  in  $V$ .

Now consider

$$T_3 = \left\{ \begin{bmatrix} 0 \\ \overline{0} \\ 0 \\ \overline{a_1} \\ a_2 \\ \overline{a_3} \\ 0 \\ 0 \end{bmatrix} \mid a_1, a_2, a_3 \in [0, 51), +, \times_n \} \subseteq V$$

is also a S-special interval pseudo linear subalgebra of  $V$  over the S-ring  $Z_{51}$ .

$T_3$  is also infinite dimensional;  $T_3$  is orthogonal to both  $T_1$  and  $T_2$  however  $T_3$  is not the orthogonal complement of  $T_1$  or  $T_2$ .

$$\text{Let } T_4 = \left\{ \left[ \begin{array}{c} \overline{a_1} \\ a_2 \\ \overline{a_3} \\ 0 \\ 0 \\ 0 \\ \overline{a_4} \\ a_5 \end{array} \right] \mid a_i \in [0, 51), 1 \leq i \leq 5, +, \times_n \} \subseteq V$$

be a S-special interval pseudo linear subalgebra of V over the S-ring  $Z_{51}$ .

We see the orthogonal complement of  $T_4$  is  $T_3$  and vice versa.

$$\text{Further } V = T_3 + T_4.$$

We can also discuss as in case of usual linear algebras notion of orthogonal complement of a set  $S \subseteq V$ .

However it is left as an exercise to find  $S^\perp = \{x \in V \mid x \times_n y = (0) \text{ for all } y \in S\}$  and prove  $S^\perp$  is a S-special vector subspace of V.

**Example 3.34:** Let

$$V = \left\{ \left( \begin{array}{c|c|c|c|c|c|c|c} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ \hline a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ \hline a_{17} & a_{18} & a_{19} & a_{20} & a_{21} & a_{22} & a_{23} & a_{24} \end{array} \right) \mid a_i \in [0, 62), \right. \\ \left. 1 \leq i \leq 24 \right\}$$

be the S-special interval pseudo linear algebra under the natural product  $\times_n$  over the S-ring  $Z_{62}$ .

We see

$$M_1 = \left\{ \left( \begin{array}{c|cc|ccc|cc} a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \mid a_i \in [0, 62), \right. \\ \left. 1 \leq i \leq 3 \right\} \subseteq V$$

is a S-special interval pseudo linear subalgebra of  $V$  over  $Z_{62}$ .

$$M_2 = \left\{ \left( \begin{array}{c|cc|ccc|cc} 0 & a_1 & a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_3 & a_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_5 & a_6 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \mid a_i \in [0, 62), \right. \\ \left. 1 \leq i \leq 6 \right\} \subseteq V$$

is again a S-special interval pseudo linear subalgebra of  $V$  over the S-ring  $Z_{62}$ .

Now

$$M_3 = \left\{ \left( \begin{array}{ccc|ccc|cc} 0 & 0 & 0 & a_1 & 0 & a_4 & 0 & 0 \\ 0 & 0 & 0 & a_2 & 0 & a_5 & 0 & 0 \\ 0 & 0 & 0 & a_3 & 0 & a_6 & 0 & 0 \end{array} \right) \mid a_i \in [0, 62), \right. \\ \left. 1 \leq i \leq 6 \right\} \subseteq V$$

is a S-special interval sublinear algebra of  $V$  over the S-ring  $Z_{62}$ .

$$\text{Clearly } M_i \cap M_j = \left\{ \left( \begin{array}{ccc|ccc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \right\} \text{ if } i \neq j, \\ 1 \leq i, j \leq 3$$

and for every  $x \in M_j$  and for every

$$y \in M_i (i \neq j; x \times_n y = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\},$$

$$1 \leq i, j \leq 3.$$

$$\text{Let } N_1 = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \mid a_i \in \{0, 0.5, 1, \right.$$

$$1.5, 2, 2.5, \dots, 60, 60.5, 61, 61.5\} \subseteq [0, 62), 1 \leq i \leq 8\} \subseteq V;$$

$N_1$  is a S-special interval quasi vector subspace of  $V$ .  $N_1$  is a S-special interval quasi vector subspace of  $V$ .

$N_1$  is not closed under product  $\times_n$ .  $N_1$  is a finite dimensional S-special quasi vector subspace of  $V$  over the S-ring  $Z_{62}$ .

Let

$$N_2 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 & a_5 & a_6 & a_7 & a_8 \end{pmatrix} \mid a_i \in \{0, 0.1, 0.2, \right.$$

$$0.3, \dots, 0.9, 1, 1.1, 1.2, \dots, 60, 1, 60.2, \dots, 60.9, 61, \\ \left. 61.1, \dots, 61.9\} \subseteq [0, 62), 1 \leq i \leq 8\} \subseteq V,$$

$N_2$  is also a S-special interval quasi vector subspace of  $V$  over the S-ring  $Z_{62}$  and  $N_2$  is of finite dimension over  $Z_{62}$ . However  $N_1$  is orthogonal with  $N_2$  and vice versa.

But  $N_1$  is not the orthogonal complement of  $N_2$  and vice versa.



In fact  $N^\perp$  of  $N_1$  is a S-special interval subspace of  $V$  over  $Z_{62}$ .

$$N^\perp = \left\{ \left( \begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ a_9 & a_{10} & a_{11} & a_{12} \end{array} \right) \mid a_i, a_j \in [0, 62), 1 \leq i, j \leq 16 \right\} \subseteq V$$

is S-special pseudo linear subalgebra of  $V$  over  $Z_{62}$ .

$$\text{We see } N_1 \cap N^\perp = \left\{ \left( \begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \right\}$$

$$\text{and } N_1 \times_n N^\perp = \left\{ \left( \begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \right\}$$

but however  $N_1 + N^\perp \neq V$ .  $N^\perp$  is the orthogonal complement of  $N_1$ .

**Example 3.35:** Let

$$V = \left\{ \left[ \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ \hline a_{13} & a_{14} & a_{15} & a_{16} \\ a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{25} & a_{26} & a_{27} & a_{28} \\ a_{29} & a_{30} & a_{31} & a_{32} \\ a_{33} & a_{34} & a_{35} & a_{36} \\ a_{37} & a_{38} & a_{39} & a_{40} \end{array} \right] \mid a_i \in [0, 69), 1 \leq i \leq 40 \right\}$$

be the S-special pseudo interval linear algebra over the S-ring  $Z_{69}$ .

$V$  has infinite and finite dimensional S-interval quasi vector subspace over the S-ring  $Z_{69}$ .

$$\text{Let } M_1 = \left\{ \left[ \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \right\} \quad a_i \in \{0, 0.5, 1, 1.5, 2, 2.5, 3, \dots,$$

$$68, 68.5\} \subseteq [0, 69), 1 \leq i \leq 4\} \subseteq V$$

be the S-special quasi vector subspace of  $V$  over the S-ring  $Z_{69}$ . Clearly  $M_1$  is a finite dimensional S-special quasi vector subspace of  $V$ .

$$\text{Let } M_2 = \left\{ \left[ \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \right\} \quad a_1, a_2, a_3, a_4 \in [0, 69), a_5, a_6,$$

$$a_7, a_8 \in \{0, 0.1, 0.2, 0.3, \dots, 0.9, 1, 1.1, 1.2, \dots, 1.9, 2, \dots, 68, 68.1, 68.2, \dots, 68.9\} \subseteq [0, 69) \subseteq V$$

be a S-special quasi vector subspace of  $V$  over the S-ring  $Z_{69}$ .

However  $M_2$  is a S-special quasi vector subspace of infinite dimensional over the S-ring  $Z_{69}$ . It is easily verified  $M_1$  is orthogonal to  $M_2$  and vice versa.

But  $M_1$  is not the orthogonal complement of  $M_2$  and  $M_2$  is not orthogonal complement of  $M_1$ .

Let

$$N_1 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline a_1 & a_2 & a_3 & a_4 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid a_i \in [0, 69); 1 \leq i \leq 64 \right\} \subseteq V$$

be the S-special pseudo interval linear subalgebra of  $V$  over the S-ring  $Z_{69}$ .

The dimension of  $N_1$  is infinite over the S-ring  $Z_{69}$ .

$$\text{Let } N_2 = \left\{ \left[ \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \\ a_5 & a_6 & a_7 & a_8 \end{array} \right] \mid a_i \in [0, 69); 1 \leq i \leq 8 \right\} \subseteq V$$

be the S-special pseudo interval linear subalgebra of  $V$  over the S-ring  $Z_{69}$ .

We see  $N_1$  is orthogonal with  $N_2$  and  $N_2$  is orthogonal with  $N_1$ .  $N_1$  is not the orthogonal complement of  $N_2$  and  $N_2$  is not the orthogonal complement of  $N_1$  and  $N_1 + N_2 \neq V$ .

**Example 3.36:** Let

$$V = \left\{ \left[ \begin{array}{c|cc|c} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ \hline a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \\ \hline a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{array} \right] \mid a_i \in [0, 55), 1 \leq i \leq 24 \right\}$$

be the S-special interval pseudo linear algebra over the S-ring  $Z_{55}$ .

$V$  has finite and infinite dimensional S-special quasi vector subspaces.

$$M_1 = \left\{ \left[ \begin{array}{c|c|c|c} a_1 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a_4 & 0 & 0 \\ \hline 0 & 0 & 0 & a_5 \end{array} \right] \mid a_i \in \{0, 0.5, 1, 1.5, 2, 2.5, \dots, 54, 54.5\}, [0, 55) \} \subseteq V$$

is a S-special quasi vector subspace of  $V$  over the S-ring  $Z_{55}$  of finite dimension

$$M_2 = \left\{ \left[ \begin{array}{c|c|c|c} 0 & 0 & 0 & a_1 \\ 0 & 0 & 0 & a_2 \\ \hline a_7 & 0 & 0 & 0 \\ a_3 & 0 & 0 & 0 \\ a_4 & 0 & 0 & 0 \\ \hline 0 & a_5 & a_6 & 0 \end{array} \right] \mid a_1, a_2, a_3, a_4 \in [0, 55), a_7, a_5, a_6 \in \{0, 0.1, 0.2, 0.3, \dots, 0.9, 1, 1.1, \dots, 54, 54.1, \dots, 54.9\} \subseteq [0, 55) \} \subseteq V$$

be the S-special quasi vector subspace of  $V$  over the S-ring  $Z_{55}$ .  $M_2$  is infinite dimensional S-quasi vector subspace of  $V$  over  $Z_{55}$ .

Let

$$N_1 = \left\{ \left[ \begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & a_1 & a_2 & 0 \\ 0 & a_3 & a_4 & 0 \\ 0 & a_5 & a_6 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right] \mid a_i \in \{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, \dots, 54, 54.2, 54.4, 54.6, 54.8\} \subseteq [0, 55) \} \subseteq V$$

$$\dots, 2, \dots, 54, 54.2, 54.4, 54.6, 54.8\} \subseteq [0, 55), 1 \leq i \leq 6\} \subseteq V$$

be a S-special quasi vector subspace of V of finite dimension over the S-ring  $Z_{55}$ .

**Example 3.37:** Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \mid a_i \in [0, 51); 1 \leq i \leq 9 \right\}$$

be a S-special interval pseudo linear algebra.

V is a non commutative S-special interval pseudo linear algebra under the product usual product  $\times$  and is commutative S-special interval linear algebra under natural product  $\times_n$ .

V has both finite and infinite dimensional S-special pseudo linear subalgebras.

$$M_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \mid a_i \in Z_5; 1 \leq i \leq 3 \right\} \subseteq V$$

is a S-special interval pseudo linear subalgebra of finite order over the S-ring  $Z_{51}$ .

$M_1$  is finite dimensional over  $Z_{51}$ .

$$M_2 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid a_i \in [0, 51); 1 \leq i \leq 2 \right\} \subseteq V$$

is a S-special pseudo interval linear subalgebra over the S-ring  $R = Z_{51}$ . Clearly dimension of  $M_2$  over R is infinite.

$$N_1 = \left\{ \left[ \begin{array}{ccc} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{array} \right] \middle| a_i \in Z_{51}; 1 \leq i \leq 9 \right\} \subseteq V \text{ is a S-special}$$

interval linear subalgebra which is non commutative over  $R = Z_{51}$ .

$N_1$  is also a finite dimensional S-special pseudo linear algebra over  $R = Z_{51}$ .

Consider

$$N_2 = \left\{ \left[ \begin{array}{ccc} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{array} \right] \middle| a_i \in Z_{51}; 1 \leq i \leq 9, \times_n \right\} \subseteq V$$

is a S-special interval pseudo linear subalgebra over  $R = Z_{51}$  which is commutative and  $N_2$  is also finite dimensional over the S-ring  $R = Z_{51}$ .

Let

$$T_1 = \left\{ \left[ \begin{array}{ccc} a_1 & a_2 & 0 \\ 0 & a_5 & a_6 \\ a_7 & 0 & a_9 \end{array} \right] \middle| a_i \in [0, 51); 1 \leq i \leq 9, \times_n \right\} \subseteq V$$

is a S-special pseudo linear subalgebra which is commutative and is of infinite dimension over  $R = Z_{51}$ .

Let

$$T_2 = \left\{ \left[ \begin{array}{ccc} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ a_3 & 0 & a_4 \end{array} \right] \middle| a_i \in [0, 51); 1 \leq i \leq 4, \times_n \right\} \subseteq V$$

be an infinite dimensional S-special linear subalgebra of  $V$  which is commutative over  $R = Z_{51}$ .

Clearly under  $\times_n$  all S-special subalgebras are commutative.

**Example 3.38:** Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{bmatrix} \mid a_i \in [0, 15); 1 \leq i \leq 25, \times \right\}$$

be the S-special interval linear algebra over the S-ring  $Z_{15} = R$ .

$V$  is non commutative pseudo linear algebra of infinite dimension over  $R = Z_{15}$ .

Let

$$P_1 = \left\{ \begin{bmatrix} a_1 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 \\ 0 & 0 & 0 & 0 & a_5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mid a_i \in [0, 15) \right\} \subseteq V$$

be the S-special quasi vector subspace of  $V$  over the S-ring  $Z_{15}$ .

Let

$$A = \begin{bmatrix} 5 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 & 5 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in P_1,$$



we find

$$A \times B = \begin{bmatrix} 5 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 7 & 5 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 10 & 1 & 0 & 0 \\ 0 & 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \notin P_1;$$

Hence  $P_1$  is only a S-special interval quasi vector subspace of  $V$  and is not a S-special linear subalgebra of  $V$  over  $Z_{15}$ .

Let

$$P_2 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{bmatrix} \mid a_i \in \{0, 0.5, 1, 1.5, 2, 2.5, \dots, 13, 13.5, 14, 14.5\} \subseteq [0, 15); 1 \leq i \leq 25 \right\} \subseteq V$$

be the S-special quasi vector subspace of  $V$  over the S-ring  $Z_{15}$ .

$P_2$  is not S-special pseudo linear subalgebra of  $V$  over  $Z_{15}$ .

$P_2$  is only finite dimensional as a S-quasi special vector subspace of  $V$ .

$$\text{Let } A = \begin{bmatrix} 0.5 & 0.5 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 1.5 & 0.5 \\ 1.5 & 0.5 & 0.5 & 0.5 & 1 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 1 & 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix} \in P_2$$

$$A \times A = \begin{bmatrix} 0.5 & 0.5 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 1.5 & 0.5 \\ 1.5 & 0.5 & 0.5 & 0.5 & 1 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 1 & 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix} \times$$

$$\begin{bmatrix} 0.5 & 0.5 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 1.5 & 0.5 \\ 1.5 & 0.5 & 0.5 & 0.5 & 1 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 1 & 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0.75 & 1 & 1.5 & 1 \\ 2 & 1.25 & 2 & 1.25 & 1.25 \\ 2.75 & 1.75 & 1.5 & 2.5 & 1.75 \\ 2.5 & 1.25 & 1.5 & 2 & 1.5 \\ 2 & 1.25 & 1.25 & 2 & 1.75 \end{bmatrix} \notin P_2.$$

Thus  $P_2$  is not a S-special linear subalgebra only a S-special quasi vector subspace of  $V$  over  $Z_{15}$ .

Now having seen examples of finite and infinite dimensional S-special quasi vector subspaces and S-special linear subalgebra both commutative we proceed onto illustrate

the notion of S-special linear transformation and S-special linear operator.

In the first place the notion of S-special linear transformation of S-special interval vector spaces and S-special interval linear algebras can be defined only if both of them are defined over the S-ring  $Z_n$ .

We will illustrate this situation by some examples.

**Example 3.39:** Let  $V = \{(a_1, a_2, a_3, a_4) \mid a_i \in [0, 6), 1 \leq i \leq 4\}$  and

$$W = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \mid a_i \in [0, 6); 1 \leq i \leq 6 \right\}$$

be two S-special interval vector spaces over the S-ring  $Z_6$ .

Let  $T : V \rightarrow W$  be a map such that

$$T \{(a_1, a_2, a_3, a_4)\} = \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \\ a_3 \\ a_4 \end{bmatrix}.$$

Clearly  $T$  is a S-special linear transformation from  $V$  to  $W$ .

We can also define  $T_1 : W \rightarrow V$  by

$$T_1 \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \right\} = (a_1 + a_2, a_3, a_4, a_5 + a_6);$$

$T_1$  is also a S-special linear transformation from  $W$  to  $V$ .

As in case of usual linear transformations we can in case of S-special linear transformation define  $\ker T_1$ .

$$\ker T_1 = \{x \in W \mid T_1(x) = (0)\} \neq (0, 0, 0, 0).$$

Thus  $\ker T_1$  is a non trivial subspace of  $W$ .

We can also define projections in case of S-special linear operations. Before we proceed to describe S-special linear projections and S-special linear operators we give some more examples of S-special linear transformations.

**Example 3.40:** Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \mid a_i \in [0, 26); 1 \leq i \leq 15 \right\}$$

and

$$W = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \end{pmatrix} \mid a_i \in [0, 26); 1 \leq i \leq 12 \right\}$$

be S-special vector spaces defined over the S-ring  $Z_{26}$ .

Define  $T : V \rightarrow W$  by

$$T\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix}\right) = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \end{pmatrix}.$$

$T$  is a  $S$ -special linear transformation from  $V$  to  $W$ .

We can define  $T_1 : W \rightarrow V$  by

$$T_1\left\{\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \end{pmatrix}\right\} = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ 0 & 0 & 0 \end{bmatrix};$$

$T_1$  is a  $S$ -special linear transformation from  $W$  to  $V$ .

Since all the  $S$ -special interval vector spaces ( $S$ -special interval pseudo linear algebras) defined over the  $S$ -ring,  $Z_n$  happens to be infinite dimensional; we have not describing results as in case of finite dimensional  $S$ -vector spaces.

**Example 3.41:** Let

$$V = \left\{ \left[ \begin{array}{ccc} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{array} \right] \mid a_i \in [0, 46); 1 \leq i \leq 18 \right\}$$

and

$$W = \left\{ \left( \begin{array}{c|cc|ccc|cc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \\ \hline a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \end{array} \right) \mid a_i \in [0, 46); 1 \leq i \leq 18 \right\}$$

be S-special interval linear algebras defined over the S-ring  $Z_{46}$ .

Define  $T : V \rightarrow W$  by

$$T \left\{ \begin{array}{ccc} a_1 & a_2 & a_3 \\ \hline a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ \hline a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{array} \right\}$$

$$= \left( \begin{array}{c|cc|ccc|cc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \\ \hline a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \end{array} \right);$$

$T$  is a S-special linear transformation.

Infact  $T$  is one to one and onto.

**Example 3.42:** Let

$$V = \left\{ \begin{bmatrix} a_1 & a_9 & a_{17} \\ a_2 & a_{10} & a_{18} \\ a_3 & a_{11} & a_{19} \\ a_4 & a_{12} & a_{20} \\ a_5 & a_{13} & a_{21} \\ a_6 & a_{14} & a_{22} \\ a_7 & a_{15} & a_{23} \\ a_8 & a_{16} & a_{24} \end{bmatrix} \mid a_i \in [0, 44); 1 \leq i \leq 24 \right\}$$

be a S-special interval vector space over the S-ring  $Z_{44}$ .

Define  $T : V \rightarrow V$  by

$$T \left( \begin{bmatrix} a_1 & a_9 & a_{17} \\ a_2 & a_{10} & a_{18} \\ a_3 & a_{11} & a_{19} \\ a_4 & a_{12} & a_{20} \\ a_5 & a_{13} & a_{21} \\ a_6 & a_{14} & a_{22} \\ a_7 & a_{15} & a_{23} \\ a_8 & a_{16} & a_{24} \end{bmatrix} \right) = \begin{bmatrix} a_1 & a_9 & a_{17} \\ 0 & 0 & 0 \\ a_3 & a_{11} & a_{19} \\ 0 & 0 & 0 \\ a_5 & a_{13} & a_{21} \\ 0 & 0 & 0 \\ a_7 & a_{15} & a_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

$T$  is a S-special linear operator on  $V$ .

$$\text{ket } T \neq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\text{Let } M_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ a_3 & 0 & 0 \\ a_4 & 0 & 0 \\ a_5 & 0 & 0 \\ a_6 & 0 & 0 \\ a_7 & 0 & 0 \\ a_8 & 0 & 0 \end{bmatrix} \mid a_i \in [0, 44); 1 \leq i \leq 8 \right\} \subseteq V;$$

the restriction of  $T$  to  $M_1$  is given by

$$T \{M_1\} = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ a_3 & 0 & 0 \\ 0 & 0 & 0 \\ a_5 & 0 & 0 \\ 0 & 0 & 0 \\ a_7 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now we can define also the notion of projection mapping.



Let

$$W_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ a_3 & 0 & 0 \\ a_4 & 0 & 0 \\ a_5 & 0 & 0 \\ a_6 & 0 & 0 \\ a_7 & 0 & 0 \\ a_8 & 0 & 0 \end{bmatrix} \mid a_i \in [0, 44); 1 \leq i \leq 8 \right\} \subseteq V$$

be the S-special vector subspace of  $V$  over the S-ring  $Z_{44}$ .

Define  $T_1 : V \rightarrow V$  by

$$T_1 \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} \\ a_{22} & a_{23} & a_{24} \end{bmatrix} \right\} \rightarrow \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ a_4 & 0 & 0 \\ a_7 & 0 & 0 \\ a_{10} & 0 & 0 \\ a_{13} & 0 & 0 \\ a_{16} & 0 & 0 \\ a_{19} & 0 & 0 \\ a_{22} & 0 & 0 \end{bmatrix} \right\}.$$

$T_1$  is a S-special linear operator on  $V$ ; infact  $T_1$  is a projection on  $V$ .

We see  $T_1 \circ T_1 = T_1$ .

Consider  $T_2 : V \rightarrow V$  given by

$$T_2 \left\{ \begin{bmatrix} a_1 & a_9 & a_{17} \\ a_2 & a_{10} & a_{18} \\ a_3 & a_{11} & a_{19} \\ a_4 & a_{12} & a_{20} \\ a_5 & a_{13} & a_{21} \\ a_6 & a_{14} & a_{22} \\ a_7 & a_{15} & a_{23} \\ a_8 & a_{16} & a_{24} \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 & a_9 & 0 \\ 0 & a_{10} & 0 \\ 0 & a_{11} & 0 \\ 0 & a_{12} & 0 \\ 0 & a_{13} & 0 \\ 0 & a_{14} & 0 \\ 0 & a_{15} & 0 \\ 0 & a_{16} & 0 \end{bmatrix} \right\}.$$

We see  $T_2$  is a S-special linear operator on  $V$  and  $T_2 \circ T_2 = T_2$ .

However if

$$W_2 = \left\{ \begin{bmatrix} 0 & a_1 & 0 \\ 0 & a_2 & 0 \\ 0 & a_3 & 0 \\ 0 & a_4 & 0 \\ 0 & a_5 & 0 \\ 0 & a_6 & 0 \\ 0 & a_7 & 0 \\ 0 & a_8 & 0 \end{bmatrix} \mid a_i \in [0, 44); 1 \leq i \leq 8 \right\} \subseteq V;$$

then  $T_2$  can be realized as a S-special linear projection of  $V$  onto  $W_2$ .

Thus we can depending on  $V$  define S-special linear projection depending on the subspaces.

Let

$$W_3 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \mid a_i \in [0, 44); 1 \leq i \leq 3 \right\} \subseteq V$$

be a S-special vector subspace of V over the S-ring  $Z_{44}$ .

Define  $T_3 : V \rightarrow V$  by

$$T_3 \left\{ \begin{bmatrix} a_1 & a_9 & a_{17} \\ a_2 & a_{10} & a_{18} \\ a_3 & a_{11} & a_{19} \\ a_4 & a_{12} & a_{20} \\ a_5 & a_{13} & a_{21} \\ a_6 & a_{14} & a_{22} \\ a_7 & a_{15} & a_{23} \\ a_8 & a_{16} & a_{24} \end{bmatrix} \right\} = \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}.$$

Clearly  $T_3$  is a S-linear operator on V.  $T_3$  is a projection of V on  $W_3$

$$T_3 \circ T_3 = T_3.$$

$$\text{Let } W_4 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \mid a_i \in [0, 44); 1 \leq i \leq 6 \right\} \subseteq V$$

be a S-special vector subspace of V over the S-ring  $R = Z_{44}$ .

Define  $T_4 : V \rightarrow V$  by

$$T_4 \left\{ \begin{bmatrix} a_1 & a_9 & a_{17} \\ a_2 & a_{10} & a_{18} \\ a_3 & a_{11} & a_{19} \\ a_4 & a_{12} & a_{20} \\ a_5 & a_{13} & a_{21} \\ a_6 & a_{14} & a_{22} \\ a_7 & a_{15} & a_{23} \\ a_8 & a_{16} & a_{24} \end{bmatrix} \right\} = \begin{bmatrix} 0 & 0 & 0 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}.$$

$T_4$  is a S-special linear operator on  $V$  and  $T_4$  is a projection of  $V$  into  $W_4$  and  $T_4 \circ T_4 = T_4$ .

Let

$$W_5 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \mid a_i \in [0, 44); 1 \leq i \leq 6 \right\} \subseteq V$$

be a S-special vector subspace of  $V$  over the S-ring  $R = Z_{44}$ .

Let

$$W_6 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_7 & a_8 & a_9 \\ 0 & 0 & 0 \end{bmatrix} \mid a_i \in [0, 44); 1 \leq i \leq 9 \right\} \subseteq V$$

be a S-special vector subspace of V over the S-ring  $R = Z_{44}$ .

Define  $T_6 : V \rightarrow V$

$$\text{by } T_6 \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} \\ a_{22} & a_{23} & a_{24} \end{bmatrix} \right\} = \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_7 & a_8 & a_9 \\ 0 & 0 & 0 \end{bmatrix}.$$

$T_6$  is a S-linear operator on V and infact  $T_6$  is a S-linear projection on V to  $W_6$ .

Define  $T_5 : V \rightarrow V$  by

$$T_5 \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} \\ a_{22} & a_{23} & a_{24} \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \right\};$$

$T_5$  is a S-linear operator on  $V$  and infact  $T_5 \circ T_5 = T_5$  and  $T_5$  is a S-linear projection on  $V$  onto  $W_5$ .

Let

$$W_7 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \quad a_i \in [0, 44); 1 \leq i \leq 6 \} \subseteq V$$

is a S-special vector subspace of  $V$  over the S-ring  $Z_{44}$ .

Define  $T_7 : V \rightarrow V$  by

$$T_7 \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} \\ a_{22} & a_{23} & a_{24} \end{bmatrix} \right\} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$T_7$  is a S linear operator on V and  $T_7$  is a S-linear projection of V onto  $W_7$ .

$$\text{We see } \ker T_i \neq \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}.$$

Thus  $\ker T_i$  is a S-special subspace of V;  $1 \leq i \leq 7$ .

**Example 3.43:** Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & \dots & \dots & \dots & a_{10} \\ a_{11} & \dots & \dots & \dots & a_{15} \\ a_{16} & \dots & \dots & \dots & a_{20} \\ a_{21} & \dots & \dots & \dots & a_{25} \end{bmatrix} \mid a_i \in [0, 93); 1 \leq i \leq 25 \right\}$$

be the S-special interval vector space over the S-ring  $Z_{93}$ .

Let

$$W_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 \\ 0 & 0 & 0 & 0 & a_5 \end{bmatrix} \mid a_i \in [0, 93); 1 \leq i \leq 5 \right\} \subseteq V$$

be a S-special vector subspace of  $V$  over the S-ring  $Z_{93}$ .

Define  $T_1 : V \rightarrow V$  by

$$T_1 \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & \dots & \dots & \dots & a_{10} \\ a_{11} & \dots & \dots & \dots & a_{15} \\ a_{16} & \dots & \dots & \dots & a_{20} \\ a_{21} & \dots & \dots & \dots & a_{25} \end{bmatrix} \right\} = \left\{ \begin{bmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 \\ 0 & 0 & 0 & 0 & a_5 \end{bmatrix} \right\}$$

$T_1$  is a S-special linear operator on  $V$ .  $T_1$  is the projection of  $V$  onto  $W_1$  and  $T_1 \circ T_1 = T_1$ .

Let

$$W_2 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mid a_i \in [0, 93); 1 \leq i \leq 5 \right\} \subseteq V$$

be a S-special vector subspace of  $V$  over the S-ring  $Z_{93}$ .



$T_2 : V \rightarrow V$  defined by

$$T_2 \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & \dots & \dots & \dots & a_{10} \\ a_{11} & \dots & \dots & \dots & a_{15} \\ a_{16} & \dots & \dots & \dots & a_{20} \\ a_{21} & \dots & \dots & \dots & a_{25} \end{bmatrix} \right\} = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\}$$

$T_2$  is a S-special linear operator on  $V$  and  $T_2$  is a S-linear projection of  $V$  onto  $W_2$ .

Further  $T_2 \circ T_2 = T_2$  is a S-special idempotent linear operator.

Let

$$W_3 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & 0 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mid a_i \in [0, 93); 1 \leq i \leq 10 \right\} \subseteq V$$

be the S-special vector subspace of  $V$  over the S-ring  $Z_{93}$ .

Define  $T_3 : V \rightarrow V$  by

$$T_3 \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & \dots & \dots & \dots & a_{10} \\ a_{11} & \dots & \dots & \dots & a_{15} \\ a_{16} & \dots & \dots & \dots & a_{20} \\ a_{21} & \dots & \dots & \dots & a_{25} \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & 0 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\};$$

$T_3$  is a S-special linear operator on  $V$ .

$T_3$  is a S-special projection operator on  $V$ .  $T_3 \circ T_3 = T_3$ .  $T_3$  is an idempotent operator on  $V$ .

Let

$$W_4 = \left\{ \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ a_1 & a_2 & \dots & a_5 \end{bmatrix} \mid a_i \in [0, 93]; 1 \leq i \leq 58 \right\} \subseteq V$$

be a S-special linear operator on  $V$ .

Define  $T_4 : V \rightarrow V$ ;

$$T_4 \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_5 \\ a_6 & a_7 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{15} \\ a_{16} & a_{17} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{25} \end{bmatrix} \right\} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 \end{bmatrix}.$$

$T_4$  is a S-linear projection of  $V$  onto  $W_4$  and  $T_4 \circ T_4$  so  $T_4$  is an idempotent operator on  $V$ .

$$\text{For } T_4 \circ T_4 \left( \begin{bmatrix} a_1 & a_2 & \dots & a_5 \\ a_6 & a_7 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{15} \\ a_{16} & a_{17} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{25} \end{bmatrix} \right)$$

$$= T_4 \left[ T_4 \left( \begin{bmatrix} a_1 & a_2 & \dots & a_5 \\ a_6 & a_7 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{15} \\ a_{16} & a_{17} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{25} \end{bmatrix} \right) \right]$$

$$= T_4 \left( \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 \end{bmatrix}.$$

Hence  $T_4 \circ T_4 = T_4$ .

We see  $T_2 \circ T_4 = (\theta)$  where  $\theta$  denotes the zero transformation on  $V$ .

$$\text{Consider } T_2 \circ T_4 \left( \begin{bmatrix} a_1 & a_2 & \dots & a_5 \\ a_6 & a_7 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{15} \\ a_{16} & a_{17} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{25} \end{bmatrix} \right)$$

$$= T_4 \left[ T_2 \left( \begin{bmatrix} a_1 & a_2 & \dots & a_5 \\ a_6 & a_7 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{15} \\ a_{16} & a_{17} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{25} \end{bmatrix} \right) \right]$$

$$= T_4 \left( \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus  $T_2 \circ T_4 (X) = (0)$  for all  $X \in V$ . Hence  $T_2 \circ T_4$  is the zero S-special linear operator on  $V$ .

$$\text{Consider } T_4 \circ T_2 \left( \begin{bmatrix} a_1 & a_2 & \dots & a_5 \\ a_6 & a_7 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{15} \\ a_{16} & a_{17} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{25} \end{bmatrix} \right)$$

$$= T_2 \left( T_4 \left( \begin{bmatrix} a_1 & a_2 & \dots & a_5 \\ a_6 & a_7 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{15} \\ a_{16} & a_{17} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{25} \end{bmatrix} \right) \right)$$

$$= T_2 \left( \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ a_1 & a_2 & \dots & a_5 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus  $T_4 \circ T_2 = \theta$  is the S-special zero linear operator on  $V$  as  $T_4 \circ T_2 (X) = (0)$  for all  $X \in V$ . Inview of all these we have the following nice theorem.

**THEOREM 3.4:** *If  $T : V \rightarrow V$  is a S-linear operator on  $V$  and if  $W_1 + \dots + W_n = V$  is the direct sum then we have*

- (i)  $T_i : V \rightarrow V$  such that  
 $T_i(V) = W_i, 1 \leq i \leq n.$
- (ii)  $T_i \circ T_j = T_j \circ T_i$  (zero operator on  $V$ )  
 $i \neq j, 1 \leq i, j \leq n.$
- (iii)  $T_i \circ T_i = T_i$  is the  $S$ -linear idempotent operator  
on  $V$ ;  $1 \leq i \leq n.$

Proof is direct and hence left as an exercise to the reader.

Next we give examples of  $S$ -special interval polynomial rings.

**Example 3.44:** Let

$$V = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in [0, 26) \right\}$$

be the  $S$ -special vector space over the  $S$ -ring  $R = Z_{26}$ .  
Dimension of  $V$  over  $R = Z_6$  is infinite.

**Example 3.45:** Let

$$V = \left\{ \sum_{i=0}^{10} a_i x^i \mid a_i \in [0, 14) \right\}$$

be the  $S$ -special vector space over the  $S$ -ring  $R = Z_{14}$ .  $V$  is of  
infinite dimension over  $R = Z_{14}$ .

Clearly  $V$  is a  $S$ -special vector space over the  $S$ -ring which  
is not a  $S$ -special linear algebra over the  $S$ -ring  $Z_{14}$ .

For if  $p(x) = a_0 + a_1x + a_2x^9$  and  $q(x) = b_0 + b_1x^8 \in V$ ;  
where  $a_0, a_1, a_2, b_0$  and  $b_1 \in [0, 14)$ .

$$\begin{aligned} p(x) \times q(x) &= a_0 + a_1x + a_2x^9 \times (b_0 + b_1x^8) \\ &= a_0b_0 + a_1b_0x + a_2b_0x^9 + a_0b_1x^8 + a_1b_1x^9 + a_2b_1x^{17} \notin V. \end{aligned}$$

Hence the claim.

**Example 3.46:** Let

$$V = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in [0, 39) \right\}$$

be the S-special interval linear algebra over the S-ring  $Z_{39} = R$ .  
 $V$  has S-quasi subspaces and S-subalgebras.

$$W_1 = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in [0, 39) \right\} \subseteq V$$

is a S-quasi subalgebra of  $V$  of infinite dimension over  $R = Z_{39}$ .

$B = \{1, x, x^2, \dots, x^n, \dots\} \subseteq W_1$  is a basis of  $W_1$  over  
 $R = Z_{39}$ .

However  $B$  is not a basis of  $V$  over  $Z_{39}$ .

Let

$$W_2 = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in \{0, 0.5, 1, 1.5, 2, 2.5, 3, \dots, 38, 38.5\} \subseteq [0, 39) \right\} \subseteq V;$$

$W_2$  is a S-special quasi vector subspace of  $V$  over the ring  
 $R = Z_{39}$ .

Clearly  $W_2$  is not a S-special linear subalgebra as if

$p(x) = 0.5x$  and  $q(x) = 4.5x^7 \in W_2$ ;

then  $p(x) \times q(x) = 0.5x \times 4.5x^7 = 2.25x^8 \notin W_2$ .

Thus  $W_2$  is only a S-special quasi vector subspace of  $V$  of  
infinite dimension over  $Z_{39}$ .

Let  $W_3 = \left\{ \sum_{i=0}^5 a_i x^i \mid a_i \in \{0, 0.1, 0.2, 0.3, \dots, 0.9, 1, 1.1, 1.2, \dots, 1.9, 2, \dots, 38.1, 38.2, \dots, 38.9\} \subseteq [0, 39); 0 \leq i \leq 5 \right\} \subseteq V$  be the S-special quasi vector subspace of  $V$  of finite dimension over  $Z_{39}$ .

It is important to observe that there is no S-special linear subalgebra of finite dimension over  $R = Z_{39}$ .

**Example 3.47:** Let

$$V = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in [0, 55) \right\}$$

be the S-special linear algebra over the S-ring  $R = Z_{55}$ .  $V$  is infinite dimensional linear algebra.  $V$  has no finite linear subalgebra.

$V$  has no finite dimensional S-special linear subalgebras.  $V$  has finite dimensional S-special quasi vector subspaces as well as infinite dimensional S-vector subspaces.

We can using these S-polynomials special interval rings build S-matrix polynomials vector spaces.

This structure is exhibited by an example or two.

**Example 3.48:** Let

$$V = \{(a_1, a_2, a_3) \mid a_i \in \left\{ \sum_{i=0}^{\infty} b_i x^i \mid b_j \in [0, 22), 1 \leq i \leq 3 \right\}\}$$

be the S-special matrix polynomial interval vector space (linear algebra) over the S-ring  $R = Z_{22}$ .

$V$  has both finite and infinite dimensional  $S$ -special quasi vector subspaces but only infinite dimensional  $S$ -special sublinear algebras.

**Example 3.49:** Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix} \mid a_i \in \left\{ \sum_{i=0}^{\infty} b_i x^i \mid b_j \in [0, 34), 1 \leq i \leq 9 \right\} \right\}$$

be the  $S$ -special interval polynomial linear algebra over the  $S$ -ring  $Z_{34}$ .

$V$  is infinite dimensional.  $V$  has finite and infinite dimensional  $S$ -special quasi vector subspaces. However all  $S$ -special linear subalgebras of infinite order.

**Example 3.50:** Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ \vdots & \vdots & \vdots & \vdots \\ a_{37} & a_{38} & a_{39} & a_{40} \end{bmatrix} \mid a_i \in \left\{ \sum_{i=0}^{\infty} b_i x^i \mid b_j \in [0, 55), 1 \leq i \leq 40 \right\} \right\}$$

be a  $S$ -special interval polynomial matrix vector space (linear algebra) over the  $S$ -ring  $Z_{55}$ .

$V$  has finite and infinite dimensional  $S$ -special.

All  $S$ -special sublinear algebras are of infinite dimension over  $R = Z_{55}$ .

It is important to note the following.



1. The concept of eigen values and eigen vectors in general cannot always be defined for S-special linear operator spaces. As the eigen values may not be always in the S-ring  $Z_n$ .
2. The concept of S-special inner product cannot be always true for the inner product may not belong to  $Z_n$ .
3. The notion of S-special linear functionals will not find its values in  $Z_n$ .

So to over come all the draw backs we are forced to define the notion of Smarandache strong special interval vector space (linear algebra) or strong Smarandache special interval vector space (linear algebra) in the following chapter.

Other than these all the properties enjoyed by the usual vector spaces is enjoyed by S-special interval vector spaces (linear algebras). The advantage of using this new notion is that when we study vector spaces over  $Z_p$ ,  $p$  a prime why not over the S-ring,  $Z_n$ .

We suggest the following problems for this chapter.

### Problems:

1. Obtain some special features enjoyed by S-special interval vector spaces over the S-ring.
2. Can S-special interval vector space over the S-ring  $Z_n$  be finite dimensional?
3. Let  $V = ([0, 35), +)$  be the special interval vector space over the S-ring  $Z_{35}$ .
  - (i) Find S-subspaces of  $V$  over  $Z_{35}$ .
  - (ii) Can  $V$  have finite S-vector subspaces over  $Z_{35}$ ?
  - (iii) How many finite dimensional S-vector subspaces are there in  $V$  over  $Z_{35}$ ?
  - (iv) Is  $V$  finite dimensional over  $Z_{35}$ ?

- (v) Is it possible to write  $V$  as a direct sum of  $S$ -subspaces?
4. Let  $V = \{[0, 46), +\}$  be the special interval vector space over the  $S$ -ring  $R = \mathbb{Z}_{46}$ .

Study questions (i) to (v) of problem 3 for this  $V$ .

5. Let  $V = \{[0, 69), +\}$  be the special interval vector space over the  $S$ -ring  $R = \mathbb{Z}_{69}$ .

Study questions (i) to (v) of problem 3 for this  $V$ .

6. Distinguish between the  $S$ -special interval vector spaces and special interval vector spaces.
7. Let  $V = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in [0, 58), 1 \leq i \leq 5\}$  be the  $S$ -special interval vector space over the  $S$ -ring  $R = \mathbb{Z}_{58}$ .

- (i) Prove  $V$  is infinite dimensional over  $R = \mathbb{Z}_{58}$ .
- (ii) Find all subspaces which are finite dimensional over  $R$ .
- (iii) Find all infinite dimensional  $S$ -special vector subspaces of  $V$  over  $R$ .
- (iv) Can  $V$  have infinite number of finite dimensional  $S$ -special vector subspaces?
- (v) Write  $V$  as a direct sum.
- (vi) Give an example of a subset  $S$  in  $V$  and its orthogonal part  $S^\perp$ .  
Prove  $S^\perp$  is a  $S$ -special interval subspace of  $V$ .
- (vii) Show in general if  $W$  is a subspace of  $V$ ,  $M$  orthogonal to  $W$  need not in general be the orthogonal complement of  $W$  in  $V$ .
- (viii) In how many ways can we write  $V$  as a direct sum of subspaces?

$$8. \quad \text{Let } V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \end{bmatrix} \mid a_i \in [0, 34); 1 \leq i \leq 10 \right\} \text{ be the S-special}$$

interval vector subspace of  $V$  over the S-ring  $R = Z_{34}$ .

Study questions (i) to (viii) of problem 7 for this  $V$ .

$$9. \quad \text{Let } V = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{30} \\ a_{31} & a_{32} & \dots & a_{40} \end{bmatrix} \mid a_i \in [0, 96); 1 \leq i \leq 40 \right\} \text{ be}$$

the S-special interval vector subspace of  $V$  over the S-ring  $R = Z_{96}$ .

Study questions (i) to (viii) of problem 7 for this  $V$ .

$$10. \quad \text{Let } V = \{[0, 52), +, \times\} \text{ be a S-special interval linear algebra over the S-ring } Z_{52}.$$

- (i) Show  $V$  is of infinite dimensional over  $Z_{52}$ .
- (ii) Show  $V$  has atleast one finite dimensional S-special interval linear subalgebra.
- (iii) Show  $V$  has both finite and infinite dimensional S-special quasi vector subspaces over  $Z_{52}$ .

11. Let  $V = \{(a_1, a_2, a_3, a_4) \mid a_i \in [0, 42), 1 \leq i \leq 4\}$  be the S-special interval linear algebra.

Study questions (i) to (iii) of problem 10 for this  $V$ .

12. Let  $V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix} \mid a_i \in [0, 122); 1 \leq i \leq 10 \right\}$  be the S-special

interval linear algebra.

Study questions (i) to (iii) of problem 10 for this  $V$ .

13. Let  $V = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_9 \\ a_{10} & a_{11} & \dots & a_{18} \\ a_{19} & a_{20} & \dots & a_{27} \end{bmatrix} \mid a_i \in [0, 77); 1 \leq i \leq 27 \right\}$  be

the S-special interval linear algebra over the S-ring  $Z_{77}$ .

Study questions (i) to (iii) of problem 10 for this  $V$ .

14. Let  $V = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_8 \\ a_9 & a_{10} & \dots & a_{16} \\ \vdots & \vdots & \dots & \vdots \\ a_{57} & a_{58} & \dots & a_{64} \end{bmatrix} \mid a_i \in [0, 46); 1 \leq i \leq 64 \right\}$  be

the S-special interval vector subspace of  $V$  over the S-ring  $R = Z_{46}$ .

Study questions (i) to (iii) of problem 10 for this  $V$ .

15. What is the algebraic structure enjoyed by the S-special interval linear operators on  $V$ ; the S-special interval linear algebra?

16. Let  $V = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{30} \end{bmatrix} \mid a_i \in [0, 39); 1 \leq i \leq 30 \right\}$  be

the S-special interval linear algebra over the S-ring  $Z_{39}$ .

Find the algebraic structure enjoyed by  $\text{Hom}_{Z_{39}}(V, V)$ .

17. Let  $V = \{(a_1, a_2, \dots, a_{10}) \mid a_i \in [0, 46), 1 \leq i \leq 10\}$  and

$W = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{bmatrix} \mid a_i \in [0, 46); 1 \leq i \leq 12 \right\}$  be two S-special

interval vector spaces over the S-ring  $Z_{46}$ .

Find the algebraic structure enjoyed by  $\text{Hom}_{Z_{46}}(V, W)$ .

18. Write  $W$  in problem 17 as direct sum of S-special interval pseudo sublinear algebras.

19. Let  $V = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_7 \\ a_8 & a_9 & \dots & a_{14} \\ \vdots & \vdots & \dots & \vdots \\ a_{43} & a_{44} & \dots & a_{49} \end{bmatrix} \mid a_i \in [0, 21); 1 \leq i \leq 49 \right\}$  be

the S-special pseudo interval linear algebra under the usual product ' $\times$ '.

- (i) Prove  $V$  is a non commutative S-special linear algebra.
- (ii) Find S-special interval quasi vector subspaces of  $V$  which are finite dimensional over  $Z_{21}$ .
- (iii) Find S-special interval quasi vector subspaces of  $V$  over  $Z_{21}$  of infinite dimension over  $Z_{21}$ .
- (iv) Does  $V$  contain finite dimensional S-interval sublinear algebras?

20. Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \mid a_i \in [0, 62), 1 \leq i \leq 16, +, \times_n \right\}$$

be the S-special interval linear algebra over the S-ring  $Z_{62}$ .

Study questions (i) to (iv) of problem 19 for this  $V$ .

$$21. \text{ Let } V_1 = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_{18} \\ a_{19} & a_{20} & \dots & a_{36} \\ a_{37} & a_{38} & \dots & a_{54} \end{bmatrix} \mid a_i \in [0, 46); 1 \leq i \leq 54 \right\}$$

be the S-special interval linear algebra over the S-ring  $Z_{46}$ .

Study questions (i) to (iv) of problem 19 for this  $V$ .

- 22. (i) Study  $\text{Hom}_{Z_{46}}(V, V_1)$ .
- (ii) Is  $\text{Hom}_{Z_{46}}(V, V_1)$ , a S-special interval vector space over the S-ring  $Z_{46}$ ?

23. Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_7 \\ a_8 & a_9 & a_{10} & \dots & a_{14} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{43} & a_{44} & a_{45} & \dots & a_{49} \end{bmatrix} \mid a_i \in [0, 111), 1 \leq i \leq 49 \right\}$$

be the S-special interval linear algebra over the S-ring  $R = Z_{41}$ .

- (i) Study questions (i) to (iv) of problem 19 for this  $V$ .
- (ii) Is  $\text{Hom}_{Z_{11}}(V, V)$  a S-special interval linear algebra over the S-ring  $Z_{111}$ ?

24. Let  $V = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_j \in [0, 55) \right\}$  be the S-special interval

linear algebra over the S-ring  $Z_{55}$ .

Study questions (i) to (iv) of problem 19 for this  $V$ .

## Chapter Four

# SMARANDACHE STRONG SPECIAL PSEUDO INTERVAL VECTOR SPACES

In this chapter we proceed onto define develop and describe the notion of Smarandache Strong Special interval pseudo vector spaces (linear algebra) denoted by SSS-interval vector space or SSS interval pseudo linear algebra. Throughout this chapter  $[0, n)$  is a special interval pseudo ring which is always taken as a S-ring.

**DEFINITION 4.1:** *Let  $V$  be a S-special interval vector space over the S-special pseudo interval ring  $[0, n)$  then we define  $V$  to be a Smarandache Special Strong pseudo interval vector space (SSS-interval vector space) over the S-special pseudo interval ring  $[0, n)$ .*

We will illustrate this situation by some simple examples.

**Example 4.1:** Let  $V = \{[0, 7) \times [0, 7)\}$  be a SSS-interval pseudo vector space over the S-special pseudo interval S-ring  $R = [0, 7)$ .

**Example 4.2:** Let  $V = \{[0, 26)\}$  be the SSS-interval pseudo vector space over the S-special pseudo interval ring  $[0, 26)$ .



**Example 4.3:** Let

$V = \{(a_1, a_2, a_3, a_4, a_5, a_6) \mid a_i \in [0, 21), 1 \leq i \leq 6\}$  be the SSS-interval pseudo vector space over the S-special pseudo interval ring  $R = [0, 21)$ .

**Example 4.4:** Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix} \mid a_i \in [0, 17); 1 \leq i \leq 10 \right\}$$

be the SSS-interval pseudo vector space over the S-special interval pseudo ring  $R = [0, 17)$ .

**Example 4.5:** Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \mid a_i \in [0, 33); 1 \leq i \leq 33 \right\}$$

be the SSS-interval pseudo vector space over the S-special interval pseudo ring  $R = [0, 33)$ .

**Example 4.6:** Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_{12} \\ a_{13} & a_{14} & \dots & a_{24} \\ a_{25} & a_{26} & \dots & a_{36} \\ a_{37} & a_{38} & \dots & a_{48} \end{bmatrix} \mid a_i \in [0, 62); 1 \leq i \leq 48 \right\}$$

be the SSS-interval pseudo vector space over the S-special interval pseudo ring  $R = [0, 62)$ .

**Example 4.7:** Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_7 \\ a_8 & a_9 & \dots & a_{14} \\ a_{15} & a_{16} & \dots & a_{21} \\ a_{37} & a_{38} & \dots & a_{28} \\ a_{29} & a_{30} & \dots & a_{35} \\ a_{36} & a_{37} & \dots & a_{42} \\ a_{43} & a_{44} & \dots & a_{49} \end{bmatrix} \mid a_i \in [0, 43); 1 \leq i \leq 49 \right\}$$

be the SSS-interval pseudo vector space over the S-special interval pseudo ring  $R = [0, 43)$ .

We can define the concept of SSS- pseudo interval vector subspace and SSS-dimension of a SSS-vector space.

We will illustrate this by the following examples.

**Example 4.8:** Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} \mid a_i \in [0, 13); 1 \leq i \leq 8 \right\}$$

be the SSS-interval pseudo vector space over the S-special pseudo interval ring  $R = [0, 13)$ .

$$W_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \mid a_1 \in [0, 13) \right\} \subseteq V$$

is a SSS-interval pseudo vector subspace of  $V$  over  $R = [0, 13)$ .

$$W_2 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \mid a_i \in [0, 13), 1 \leq i \leq 3 \right\} \subseteq V$$

is a SSS-interval pseudo vector subspace of  $V$  over  $R = [0, 13)$ .

$$W_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \mid a_i \in [0, 13), 1 \leq i \leq 3 \right\} \subseteq V$$

is a SSS-interval vector pseudo subspace of  $V$  over the S-special interval pseudo ring  $R = [0, 13)$ .

$$W_4 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ a_2 \\ 0 \\ a_3 \\ 0 \\ a_4 \\ 0 \end{bmatrix} \mid a_i \in [0, 13), 1 \leq i \leq 4 \right\} \subseteq V$$

is again a SSS-interval pseudo vector subspace of  $V$  over  $R = [0, 13)$ .

**Example 4.9:** Let

$V = \{(a_1, a_2, a_3, \dots, a_{15}) \mid a_i \in [0, 22), 1 \leq i \leq 15\}$  be the SSS-interval pseudo vector space over the S-special pseudo interval ring  $R = [0, 22)$ .

$W_1 = \{(a_1, a_2, a_3, 0, \dots, 0) \mid a_i \in [0, 22), 1 \leq i \leq 3\} \subseteq V$  is a SSS-interval pseudo vector subspace of  $V$  over  $R = [0, 22)$ .

$W_2 = \{(0, 0, 0, 0, 0, a_1, a_2, a_3, a_4, a_5, 0, 0, 0, 0, 0) \mid a_i \in [0, 22), 1 \leq i \leq 5\} \subseteq V$  is a SSS-interval pseudo vector subspace of  $V$  over the ring  $R = [0, 22)$ .

$W_3 = \{(0, 0, \dots, 0, a_1, a_2) \mid a_1, a_2 \in [0, 22)\} \subseteq V$  is a SSS-interval vector pseudo subspace of  $V$  over the ring  $R = [0, 22)$ .

**Example 4.10:** Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix} \mid a_i \in [0, 6), 1 \leq i \leq 30 \right\}$$

be the SSS-interval pseudo vector space over the S-special pseudo interval ring  $R = [0, 6)$ .

$$W_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ a_4 & a_5 & a_6 \end{bmatrix} \mid a_i \in [0, 6), 1 \leq i \leq 6 \right\} \subseteq V$$

is the SSS-interval pseudo vector space over the S-special pseudo interval ring  $R = [0, 6)$ .

$$W_2 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \mid a_i \in [0, 26), 1 \leq i \leq 9 \right\} \subseteq V$$

is the SSS-interval pseudo vector space over the S-special pseudo interval ring  $R = [0, 6)$ .

**Example 4.11:** Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix} \mid a_i \in [0, 43), 1 \leq i \leq 9 \right\}$$

be the SSS-linear pseudo algebra over the S- pseudo ring  $R = [0, 43)$ .

$V$  is finite dimensional over  $R$ .  $V$  has several SSS-linear pseudo subalgebras of  $V$  over the  $S$ -ring  $R = [0, 43)$ .

$$W_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \mid a_1 \in [0, 43) \right\} \subseteq V$$

$$W_2 = \left\{ \begin{bmatrix} 0 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \mid a_2 \in [0, 43) \right\} \subseteq V \text{ and so on.}$$

$$W_9 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ a_9 \end{bmatrix} \mid a_9 \in [0, 43) \right\} \subseteq V$$

are the nine SSS-interval linear pseudo subalgebras of  $V$  each of dimension one over  $R = [0, 43)$ .

$$M_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \mid a_1, a_2 \in [0, 43) \right\} \subseteq V,$$

$$M_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ a_1 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \mid a_1, a_2 \in [0, 43) \right\} \subseteq V,$$

$$M_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a_1, a_2 \in [0, 43) \right\} \subseteq V \text{ and}$$

$$M_4 = \left\{ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \mid a_i \in [0, 43), 1 \leq i \leq 3 \right\} \subseteq V$$

are the four SSS-linear pseudo subalgebras of  $V$  over the  $S$ -ring  $[0, 43)$ .

$M_1$ ,  $M_2$  and  $M_3$  are two dimension SSS-linear pseudo subalgebras of  $V$  over  $R = [0, 43)$ .

$M_4$  is of dimension three over  $R = [0, 43)$ .

Let

$$P_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \mid a_i \in [0, 43), 1 \leq i \leq 4 \right\} \subseteq V,$$

be the SSS linear pseudo subalgebra of dimension four over the S-ring  $R = [0, 43)$ .

$$P_2 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \mid a_i \in [0, 43), 1 \leq i \leq 5 \right\} \subseteq V$$

is a SSS-linear pseudo subalgebra of dimension five over  $[0, 43)$ .



$$P_3 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ a_2 \\ a_3 \\ a_4 \\ 0 \\ a_5 \\ a_6 \\ 0 \end{bmatrix} \mid a_i \in [0, 43), 1 \leq i \leq 6 \right\} \subseteq V$$

is a SSS-linear pseudo subalgebra of dimension six over the S-ring  $[0, 43)$ .

$$P_4 = \left\{ \begin{bmatrix} 0 \\ a_1 \\ a_2 \\ \vdots \\ a_7 \\ 0 \end{bmatrix} \mid a_i \in [0, 43), 1 \leq i \leq 7 \right\} \subseteq V$$

is a SSS-linear pseudo subalgebra of dimension seven over the S-special pseudo ring.

$$P_5 = \left\{ \begin{bmatrix} 0 \\ a_1 \\ \vdots \\ 0 \\ a_8 \end{bmatrix} \mid a_i \in [0, 43), 1 \leq i \leq 8 \right\} \subseteq V$$

is a SSS- pseudo linear algebra of dimension eight over the S-pseudo ring.

The SSS- pseudo linear algebra which is of dimension nine over the S-ring  $R = [0, 43)$  has SSS-linear pseudo subalgebras of all dimensions between one and eight.

**Example 4.12:** Let

$$V = \left\{ \begin{bmatrix} a_1 & a_5 & a_9 \\ a_2 & a_6 & a_{10} \\ a_3 & a_7 & a_{11} \\ a_4 & a_8 & a_{12} \end{bmatrix} \mid a_i \in [0, 29), 1 \leq i \leq 12, +, \times_n \right\}$$

be the SSS-linear pseudo algebra over the S-ring  $R = [0, 29)$ .

Dimension of  $V$  over  $R$  is 12.  $V$  has SSS-linear pseudo subalgebra of various dimensions.

$V$  is a usual vector space (linear algebra) over the field  $Z_{29} \subseteq [0, 29)$ .

Several interesting properties can be derived. However this  $V$  has no SSS quasi pseudo vector subspaces.

$V = W_1 + W_2 + W_3$  where

$$W_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ a_3 & 0 & 0 \\ a_4 & 0 & 0 \end{bmatrix} \mid a_i \in [0, 29), 1 \leq i \leq 4 \} \subseteq V,$$

$$W_2 = \left\{ \begin{bmatrix} 0 & a_1 & 0 \\ 0 & a_2 & 0 \\ 0 & a_3 & 0 \\ 0 & a_4 & 0 \end{bmatrix} \mid a_i \in [0, 29), 1 \leq i \leq 4 \} \subseteq V \text{ and}$$

$$W_3 = \left\{ \begin{bmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ 0 & 0 & a_3 \\ 0 & 0 & a_4 \end{bmatrix} \mid a_i \in [0, 29), 1 \leq i \leq 4 \right\} \subseteq V$$

are SSS-linear pseudo subalgebra and  $V = W_1 + W_2 + W_3$ .

Let

$$M_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid a_i \in [0, 29), 1 \leq i \leq 4 \right\} \subseteq V,$$

$$M_2 = \left\{ \begin{bmatrix} 0 & 0 & a_1 \\ a_3 & a_4 & a_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid a_i \in [0, 29), 1 \leq i \leq 4 \right\} \subseteq V,$$

$$M_3 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_1 & a_2 \\ a_3 & a_4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid a_i \in [0, 29), 1 \leq i \leq 4 \right\} \subseteq V,$$

$$M_4 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ a_4 & 0 & 0 \end{bmatrix} \mid a_i \in [0, 29), 1 \leq i \leq 4 \right\} \subseteq V \text{ and}$$

$$M_5 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_1 \\ a_2 & a_3 & a_4 \end{bmatrix} \mid a_i \in [0, 29), 1 \leq i \leq 4 \right\} \subseteq V$$

are SSS-interval pseudo linear subalgebras of  $V$  over  $R = [0, 29)$ .

We see

$$M_i \cap M_j \neq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ if } i \neq j, 1 \leq i, j \leq 5.$$

Thus  $V \subseteq M_1 + M_2 + M_3 + M_4 + M_5$ . Also  $M_i$  is not orthogonal to any one of the  $M_j$ 's;  $i \neq j$ . We see  $W_1$  is the orthogonal to  $W_2$  but  $W_2$  is not the orthogonal complement of  $W_1$ .

$$\text{Let } P_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid a_i \in [0, 29), 1 \leq i \leq 5 \right\} \subseteq V$$

be the SSS-interval linear pseudo subalgebra of  $V$  over the S-special interval ring  $R = [0, 29)$ .

$$P_2 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a_1 \\ a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 \end{bmatrix} \mid a_i \in [0, 29), 1 \leq i \leq 7 \right\} \subseteq V$$

is the SSS-interval pseudo linear subalgebra of  $V$  over the ring  $R = [0, 29)$ .

We see  $P_1$  is the orthogonal complement of  $P_2$  and vice versa.

Further  $P_1 + P_2 = V$  and

$$P_1 \cap P_2 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \text{ so } P_1 \text{ and } P_2 \text{ gives the direct sum of } V.$$

Now having seen examples of SSS-interval pseudo linear algebras direct sum and SSS-interval pseudo sublinear algebras; we now proceed onto define only special properties and we are not interested in studying other details.

We are more interested in other properties which we are not in a position to impose in case of S-special pseudo interval linear algebras.

Let  $V$  be a SSS-linear pseudo algebra over the S-ring  $R = [0, n)$ .

On  $V$  we define the notion of pseudo inner product for if  $x, y \in V$   $\langle x, y \rangle$  is the pseudo inner product  $\langle x, y \rangle : V \rightarrow R$ ; we see  $\langle x, y \rangle = 0$  even if  $x \neq 0$  and  $y \neq 0$ .  $x, y \in V$  all other properties remain the same.

This includes  $\langle x, x \rangle = 0$  even if  $x \neq 0$ .

Thus by defining SSS-interval pseudo linear algebras  $V$ ; we can define the pseudo inner product.

We can also define on SSS-interval pseudo linear algebra the notion of SSS-eigen values, SSS-eigen vectors and SSS-characteristic polynomials.

Further we can define SSS-linear functionals using SSS-interval linear algebra  $V$ .

All these concepts will be described only by examples.

**Example 4.13:** Let

$V = \{(a_1, a_2, a_3, a_4) \text{ where } a_i \in [0, 15), 1 \leq i \leq 4\}$  be a SSS-pseudo linear algebra over the S-special interval pseudo ring  $R = [0, 15)$ .

We define SSS-linear functional on  $V$  as follows:

$f_{\text{sss}} : V \rightarrow [0, 15)$  so that  $f_{\text{sss}}$  can also be realized as a SSS-linear transformation of  $V$  to  $[0, 15)$  as  $[0, 15)$  can be realized as a SSS-pseudo vector space of dimension one over  $[0, 15)$ .

$f_{\text{sss}} : V \rightarrow [0, 15)$  is a SSS-linear functional; if  $x = (0.112, 3.001, 4.0007, 8) \in V$  define  $f_{\text{sss}}(x) = a_1 \times 0.112 + a_2 \times 3.001 + a_3 \times 4.0007 + a_4 \times 8$  where  $a_i \in [0, 15), 1 \leq i \leq 4$ .

We see if  $V^* = \{\text{Collection of all SSS-linear functionals on } V\}$  then  $V^*$  is also a SSS-interval vector space over  $[0, 15)$ .

All this study can be derived with simple and appropriate modifications. It is also left as an exercise to the reader to prove  $\dim V^* = \dim V$ .

We define SSS-annihilator of a subset  $S$  of a SSS-vector space  $V$  is the set  $S^\circ$  of SSS-linear functionals  $f_{\text{sss}}$  on  $V$  such that  $f_{\text{sss}}(\alpha) = 0$  for every  $\alpha \in S$ .

The SSS-subset  $S^\circ$  of  $V^*$  is a SSS-pseudo vector subspace of  $V$ .

The following theorems can be proved by the interested reader.

**THEOREM 4.1:** *Let  $V$  be a finite dimensional SSS-pseudo interval vector space over the S-special pseudo interval ring  $R = [0, n)$ .  $W$  be a SSS-subspace of  $V$ .*

Then  $\dim W + \dim W^\circ = \dim V$ .

**THEOREM 4.2:** Let  $V$  be a finite dimensional SSS-vector pseudo space over the  $S$ -special interval pseudo  $S$ -ring  $[0, n]$ .

For each vector  $\alpha$  in  $V$  define  $L_\alpha(f_{SSS}) = f_{SSS}(\alpha)$ ;  $f_{SSS}$  in  $V^*$ . Then the mapping  $\alpha \rightarrow L_\alpha$  is an isomorphism of  $V$  into  $V^{**}$ .

**THEOREM 4.3:** Let  $g_{SSS}, f_{SSS}^1, f_{SSS}^2, \dots, f_{SSS}^r$  be SSS-linear functionals on a SSS- pseudo vector space  $V$  with respect to the SSS-null space  $N_{SSS}, N_{SSS}^1, N_{SSS}^2, \dots, N_{SSS}^r$  respectively. Then  $g_{SSS}$  is a linear combination of  $f_{SSS}^1, f_{SSS}^2, \dots, f_{SSS}^r$  if and only if  $N_{SSS}$  contains the intersection  $N_{SSS}^1 \cap N_{SSS}^2 \cap \dots \cap N_{SSS}^r$ .

Now we can as in case of usual vector spaces define SSS-eigen values etc. Let  $A = (a_{ij})$   $n \times n$  be a  $n \times n$  matrix  $a_{ij} \in [0, m]$ ,  $1 \leq i, j \leq n$ .

The SSS pseudo characteristic value of  $A$  in  $[0, m]$  is a scalar  $c$  in  $[0, m]$  such that the matrix  $(A - CI)$  is non invertible.

$C$  is the SSS pseudo characteristic value of  $A$  if and only if  $\det(A - CI) = 0$  or equivalently  $\det(CI - A) = 0$ , we form the matrix  $(xI - A)$  with polynomial entries and consider polynomial  $f(x) = \det(xI - A)$ .

Clearly the SSS-characteristic values of  $A$  in  $[0, m]$  are just the scalars  $C$  in  $[0, m]$  such that  $f(C) = 0$ .

For this reason  $f$  is called the SSS- pseudo characteristic polynomial of  $A$ .

Here also  $f$  is monic and  $f(x) \in \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in [0, m] \right\}$ .

All properties associated with characteristic polynomials are true in case of these SSS-polynomials.

**Example 4.14:** Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_7 \end{bmatrix} \mid a_i \in [0, 6), 1 \leq i \leq 6 \right\}$$

be the SSS-interval pseudo vector space over the S-special pseudo interval ring  $R = [0, 6)$ .

Define

$$f_{\text{SSS}} : V \rightarrow [0, 6) \text{ by } f_{\text{SSS}} \left( \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_7 \end{bmatrix} \right) = a_1 + a_2 + \dots + a_7 \pmod{6}.$$

$f_{\text{SSS}}$  is a linear functional on  $V$ .

$$\text{For instance if } x = \begin{bmatrix} 3.002 \\ 4.701 \\ 3.0175 \\ 2.0016 \\ 0.90121 \\ 5.03215 \\ 1.3141 \end{bmatrix} \in V.$$



$$f_{sss}(x) = f_{sss} \left( \begin{bmatrix} 3.002 \\ 4.701 \\ 3.0175 \\ 2.0016 \\ 0.90121 \\ 5.03215 \\ 1.3141 \end{bmatrix} \right) =$$

$$3.002 + 4.701 + 3.0175 + 2.0016 + 0.90121 + 5.03215 + 1.3141 \\ = 1.96956 \in [0, 6).$$

This is the way  $f_{sss}$  is a SSS-linear functional on  $V$ .

Interested reader can form any number of such SSS-linear functionals on SSS- pseudo vector spaces over  $[0, n)$ .

**Example 4.15:** Let

$$V = \left\{ \begin{bmatrix} a_1 & \dots & a_6 \\ a_7 & \dots & a_{12} \\ a_{13} & \dots & a_{18} \\ a_{19} & \dots & a_{24} \\ a_{25} & \dots & a_{30} \\ a_{30} & \dots & a_{36} \end{bmatrix} \right\} \quad a_i \in [0, 15), 1 \leq i \leq 36, +, \times \}$$

be the SSS- pseudo interval linear algebra over the S-special interval ring  $R = [0, 15)$ .

Define  $f_{sss} : V \rightarrow [0, 15)$  by  
 $f_{sss}(A) = a_{11} + a_{22} + \dots + a_{66} \pmod{15}$  where  $A = (a_{ij}) \in V$  that  
 is  $f_{sss}(A) = \text{trace } A$ .

$f_{sss}$  is SSS-linear functional on  $V$ .

Suppose

$$A = \begin{bmatrix} 0.132 & 0 & 9 & 6 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0.92 & 0 & 1.31 & 0 & 0 & 0 \\ 7.52 & 6.3 & 0 & 4.31 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3.101 & 0 \\ 0 & 0 & 7.31 & 0 & 0 & 7.1 \end{bmatrix} \in V.$$

$$f_{\text{SSS}}(A) = \text{trace } A$$

$$\begin{aligned} &= 0.132 + 0 + 1.41 + 4.31 + 3.101 + 7.1 \\ &= 15.953 \pmod{15} \\ &= 0.953 \in [0, 15). \end{aligned}$$

$f_{\text{SSS}}$  is a SSS-linear functional on  $V$ .

Now having seen examples of SSS-linear functionals we now proceed onto define more properties of SSS-linear algebras.

**Example 4.16:** Let

$$V = \left\{ \begin{bmatrix} a_1 & \dots & a_6 \\ a_7 & \dots & a_{12} \\ a_{13} & \dots & a_{18} \\ a_{19} & \dots & a_{24} \\ a_{25} & \dots & a_{30} \\ a_{30} & \dots & a_{36} \end{bmatrix} \mid a_i \in [0, 19), 1 \leq i \leq 40 \right\}$$

be a SSS-vector space over the S-special pseudo interval ring  $R = [0, 19)$ .

Define  $f_{\text{SSS}} : V \rightarrow [0, 19)$  as

$f_{\text{sss}}(A) = \text{sum of column two} + \text{sum of column four (mod 19)}$ .

$$= (a_1 + a_6 + a_{10} + a_{14} + a_{18} + a_{22} + a_{26} + a_{30} + a_{34} + a_{38}) + (a_4 + a_8 + a_{12} + a_{16} + a_{20} + a_{24} + a_{28} + a_{32} + a_{36} + a_{40}) \pmod{19}.$$

$f_{\text{sss}}$  is a SSS-linear functional on  $V$ .

**Example 4.17:** Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{30} \end{bmatrix} \mid a_i \in [0, 23), 1 \leq i \leq 30 \right\}$$

be the SSS-linear algebra over the S-special pseudo interval ring  $R = [0, 23)$ .

Define  $f_{\text{sss}} : V \rightarrow [0, 23)$  as  $f_{\text{sss}}(A) = \text{sum of the 3}^{\text{rd}}$  row

$$= a_{11} + a_{12} + a_{13} + \dots + a_{20}.$$

$f_{\text{sss}}$  is a SSS-linear functional on  $V$ .

**Example 4.18:** Let

$$A = \begin{bmatrix} 0.001 & 0 & 2 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0.03 \end{bmatrix} \text{ with elements from } [0, 3).$$

We find the SSS- pseudo characteristic polynomial associated with  $A$ .

$$|Ix - A| = \left| \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix} - \begin{bmatrix} 0.001 & 0 & 2 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0.03 \end{bmatrix} \right|$$

$$\begin{aligned}
&= \left| \begin{bmatrix} x-0.001 & 0 & 2 \\ 0 & x-0.04 & 0 \\ 0 & 0 & x-0.03 \end{bmatrix} \right| \\
&= (x + 2.999) (x + 2.96) (x + 2.97) \\
&= 0.
\end{aligned}$$

$x = 0.001, 0.04$  and  $0.03$ .

Thus the SSS- pseudo eigen values of  $A$  are  $0.001, 0.04$  and  $0.03$ . Now we can find SSS- pseudo eigen values for any square matrix with entries from  $[0, n)$ . If the values are real we get these SSS- pseudo eigen values.

**Example 4.19:** Let  $V = \{(a_1, a_2, a_3) \mid a_i \in [0, 5), 1 \leq i \leq 3\}$  be SSS-interval pseudo vector space over the S-special pseudo interval ring  $R = [0, 5)$ .

We define

$\langle x, y \rangle_{\text{SSS}} : V \times V \rightarrow [0, 5)$  as if  $x = (0.0221, 0.31, 0.7)$  and  $y = (0.01, 0.04, 0.071) \in V$  then  $\langle x, y \rangle_{\text{SSS}} = (0.0221, 0.31, 0.7) \times (0.01, 0.04, 0.071)$

$$\begin{aligned}
&= (0.0221 \times 0.01 + 0.31 \times 0.04 + 0.7 \times 0.071) \\
&= 0.000221 + 0.0124 + 0.0497 \\
&= 0.062321 \in [0, 5).
\end{aligned}$$

This is the way the inner product is defined.

Let  $x = (2, 1, 1)$  and  $y = (1, 3, 0) \in V$ .

$$\begin{aligned}
\langle x, y \rangle_{\text{SSS}} &= \langle (2, 1, 1), (1, 3, 0) \rangle \\
&= 2 + 3 + 0 \pmod{5} \\
&= 5.
\end{aligned}$$

Thus  $x$  is orthogonal to  $y$ .

Let  $V$  be SSS-vector space.

If on  $V$  we have an inner product  $\langle x, y \rangle_{\text{sss}}$  defined then we call  $V$  to be SSS-inner product space over  $R = [0, n)$  ( $n < \infty$ ).

Now if  $V$  is a SSS- pseudo inner product space over the S-special interval pseudo ring  $R = [0, n)$ , we say for any  $W \subseteq V$ ,  $W$  the SSS- pseudo vector subspace of  $V$  the orthogonal complement  $W$  to be  $W^\perp = \{x \in V \mid \langle x, y \rangle = 0 \text{ for all } y \in W\}$ .

We will illustrate this situation by some examples.

**Example 4.20:** Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \mid a_i \in [0, 21), 1 \leq i \leq 6 \right\}$$

be a SSS-vector space over the S-special pseudo interval ring  $R = [0, 21)$ .

Let  $V$  be an inner SSS- pseudo product space where  $\langle x, y \rangle_{\text{sss}}$

$$\text{is defined by } W^\perp = \left\{ \begin{bmatrix} 0 \\ a_1 \\ 0 \\ a_2 \\ a_3 \\ 0 \end{bmatrix} \mid a_i \in [0, 21), 1 \leq i \leq 3 \right\} \subseteq V \text{ is such}$$

that for every  $x \in W$  and for  $y \in W^\perp$   $\langle x, y \rangle_{\text{sss}} = 0 \in [0, 21)$ .  
 $W^\perp = \{x \in V \mid \langle x, y \rangle_{\text{sss}} = 0 \text{ for all } y \in W\}$ .

Thus we have SSS-orthogonal vectors.

$$W \cap W^{\perp} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\text{Thus if } x = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \text{ and } y = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{bmatrix} \in V \text{ then}$$

$$\langle x, y \rangle_{\text{SSS}} = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 + a_5b_5 + a_6b_6 \pmod{21}.$$

$$\text{Let } W = \left\{ \begin{bmatrix} a_1 \\ 0 \\ a_2 \\ 0 \\ 0 \\ a_3 \end{bmatrix} \mid a_i \in [0, 21), 1 \leq i \leq 3 \right\} \subseteq V$$

be a SSS- pseudo vector subspace of  $V$  over the pseudo ring  $R = [0, 21)$ .

$$W^\perp = \left\{ \begin{bmatrix} 0 \\ a_1 \\ 0 \\ a_2 \\ a_3 \\ 0 \end{bmatrix} \mid a_i \in [0, 21), 1 \leq i \leq 3 \right\} \subseteq V$$

is the orthogonal with  $W$  with  $W \cap W^\perp = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$  answer is yes?

Now suppose  $S$  is only a subset in  $V$ ,  $V$  a SSS-inner product space.

What will be  $S^\perp$ .

$$\text{Let } S = \left\{ \begin{bmatrix} 0.7 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.5 \\ 0 \end{bmatrix} \right\} \subseteq V$$

be a subset of  $V$ .

$$\text{To find } S^\perp = \{x \in V \mid \langle x, y \rangle_{\text{sss}} = 0 \text{ for all } y \in S\}$$

$$S^\perp = \left\{ \begin{bmatrix} 0 \\ a_1 \\ a_2 \\ a_3 \\ 0 \\ a_4 \end{bmatrix} \mid a_i \in V, 1 \leq i \leq 4 \right\} \subseteq V$$

is the orthogonal to  $S$  of SSS- pseudo subspace of  $V$ .

$$\text{Clearly } S \cap S^\perp = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ but however } S + S^\perp \neq V.$$

Thus the orthogonal complement of a subset is also a SSS- pseudo subspace of  $V$ .

**Example 2.21:** Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ \vdots & \vdots & \vdots \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \mid a_i \in [0, 41), 1 \leq i \leq 15 \right\} \subseteq V$$

be the SSS- pseudo vector space of over the  $S$ -special pseudo interval ring  $R = [0, 41)$ .

$$\text{Define } \langle x, y \rangle_{\text{SSS}} : V \rightarrow [0, 41) \text{ by } \langle A, B \rangle_{\text{SSS}} = \sum_{i=1}^{15} a_i b_i \pmod{41}$$



$$\text{where } A = \begin{bmatrix} a_1 & a_2 & a_3 \\ \vdots & \vdots & \vdots \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 & b_2 & b_3 \\ \vdots & \vdots & \vdots \\ b_{13} & b_{14} & b_{15} \end{bmatrix} \in V.$$

$\langle A, B \rangle_{\text{SSS}}$  is a SSS-inner product on the SSS- pseudo vector space  $V$  over special pseudo  $R = [0, 41)$ .

$$\text{Let } A = \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \in V$$

we see there exists infinitely many  $B \in V$  such that  $\langle A, B \rangle_{\text{SSS}} = 0$ .

So even for a single element  $S = \{A\}$  we see

$$S^\perp = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ \vdots & \vdots & \vdots \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \mid a_i \in [0, 41), 1 \leq i \leq 12 \right\} \subseteq V.$$

$S^\perp$  is a SSS- pseudo vector subspace of  $V$ .  $S^\perp + S \neq V$ .

$$\text{But } S \cap S^\perp = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}.$$

Interested reader can get the analogue of the Gram Schmidt process in case of SSS- pseudo vector space with some direct and appropriate modifications.

**THEOREM 4.4:** *Let  $W$  be a SSS-finite pseudo dimensional subspace of a SSS-inner product space  $V$  over the S-special interval pseudo ring  $R = [0, n)$ . Let  $E_{SSS}$  be a SSS orthogonal projection of  $V$  on  $W$ .*

*Then  $E_{SSS}$  is an idempotent SSS-linear transformation of  $V$  onto  $W$ ,  $W^\perp$  is the SSS null space of  $E$  and  $V = W \oplus W^\perp$ .*

Proof is similar to as that of usual spaces hence left as an exercise to the reader.

Next we proceed onto define the notion of SSS- pseudo polynomial vector space over S- pseudo special interval ring  $[0, n)$ .

**DEFINITION 4.2:** *Let*

$$V = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in [0, n), n < \infty \right\}$$

*be the SSS-polynomial pseudo vector space defined over the S-special pseudo interval ring  $R = [0, n)$ .  $V$  is an infinite dimensional SSS vector space over  $R$ .  $V$  is also a SSS- pseudo linear algebra over  $R$ .*

We will first illustrate this situation by examples.

**Examples 4.22:** *Let*

$$V = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in [0, 43) \right\}$$

*be a SSS-polynomial pseudo vector space over the S-special interval S- pseudo interval ring  $R = [0, 43)$ .*

Let  $V$  be a SSS- pseudo linear algebra under the usual product of polynomials over the S- pseudo special interval ring  $R = [0, 43)$ .

Let  $W = \left\{ \sum_{i=0}^{11} a_i x^i \mid a_i \in [0, 43), 0 \leq i \leq 11 \right\} \subseteq V$  is a SSS

quasi pseudo vector subspace of  $V$  and  $W$  is not a SSS- pseudo linear subalgebra of  $V$  as product of two polynomials is not defined in  $W$ .

Only in case of SSS polynomial linear pseudo algebras alone we are in a position to define SSS- pseudo quasi vector subspaces of  $V$ .

Almost all properties associated with usual vector spaces can be extended in case of SSS-vector spaces with some appropriate modifications.

We suggest some problems for the reader.

Some of the problems are difficult at research level and some of them are simple and some are little hard and consume more time.

## Problems

1. Obtain some special features enjoyed by SSS- pseudo interval vector spaces over the S- pseudo ring  $[0, n)$  ( $n < \infty$ ).
2. Spell out some of the advantages of using SSS- pseudo interval spaces in the place of S-interval pseudo special vector spaces.
3. Is it possible to define S-linear functionals using S-special interval pseudo linear vector spaces?
4. Let  $V = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in [0, 46), 1 \leq i \leq 5\}$  be the SSS- pseudo linear algebra over the S- pseudo special interval ring  $R = [0, 46)$ .

(i) Find dimension of  $V$  over  $R$ .

- (ii) Does we have infinite number of basis for  $V$ ?
- (iii) Find all SSS- pseudo subspaces of dimension two over  $R$ .
- (iv) Find  $W$  a SSS- pseudo subspace of  $V$  so that
  - i.  $W^\perp$  is its orthogonal complement.
  - ii.  $W_1^\perp$  is just orthogonal with  $W$  and is not the orthogonal complement of  $W$ .

$$5. \quad \text{Let } W = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_5 \\ a_6 & a_7 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{15} \\ a_{16} & a_{17} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{25} \end{bmatrix} \mid a_i \in [0, 7), 1 \leq i \leq 25 \right\} \text{ be}$$

the SSS- pseudo special interval vector space (linear algebra under  $\times$  or  $\times_n$ ) over the ring  $R = [0, 7)$ .

- (i) Study questions (i) to (iv) of problem 4 for this  $V$ .
  - (ii) Prove under  $\times$   $V$  is a non commutative SSS- pseudo linear algebra.
6. Let  $M = \{(a_1, a_2, \dots, a_{11}) \mid a_i \in [0, 18), 1 \leq i \leq 11\}$  be a SSS- pseudo linear algebra over the S-special interval ring  $R = [0, 18)$ .

$$7. \quad \text{Let } P = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{15} \end{bmatrix} \mid a_i \in [0, 43); 1 \leq i \leq 15 \right\} \text{ be a SSS-}$$

pseudo linear algebra over the S-special interval ring  $R = [0, 43)$ .

- (i) Study questions (i) to (iv) of problem 4 for this S.
- (ii) Find the algebraic structure enjoyed by  $\text{Hom}_R(V, V)$ .

8. Let  $V = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_9 \\ a_{10} & a_{11} & \dots & a_{18} \\ a_{19} & a_{20} & \dots & a_{27} \\ \vdots & \vdots & \vdots & \vdots \\ a_{73} & a_{74} & \dots & a_{81} \end{bmatrix} \mid a_i \in [0, 46]; 1 \leq i \leq 81 \right\}$

be a SSS-non commutative pseudo linear algebra over the S-special pseudo interval ring  $R = [0, 46]$ .

- (i) Study questions (i) to (iv) of problem 4 for this S.
  - (ii) What is the distinct feature enjoyed by V as V is a SSS-non commutative linear algebra?
9. Let  $V = \{(a_1, a_2, \dots, a_{10}) \mid a_i \in [0, 23]\}$  be a SSS- pseudo linear algebra over the S-special pseudo interval ring  $R = [0, 23]$ .
- (i) How many distinct inner products be defined on V?
  - (ii) What is the dimension of V as a SSS- pseudo vector space over R?
  - (iii) What is the dimension of V as a SSS- pseudo linear algebra over R?
  - (iv) In how many ways can V be written as a direct sum?

$$10. \quad \text{Let } V = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \\ a_{13} & a_{14} \\ a_{15} & a_{16} \\ a_{17} & a_{18} \end{bmatrix} \mid a_i \in [0, 48); 1 \leq i \leq 18 \right\} \text{ be SSS-}$$

pseudo linear algebra over the S-special pseudo interval ring  $R = [0, 48)$  under the natural product  $\times_n$  of matrices.

Study questions (i) to (iv) of problem 9 for this  $V$ .

$$11. \quad \text{Let } V = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \end{pmatrix} \mid a_i \in [0, 35); \right.$$

$1 \leq i \leq 10 \}$  be a SSS- pseudo linear algebra over the S- pseudo ring  $R = [0, 35)$ .

Study questions (i) to (iv) of problem 9 for this  $V$ .

12. Describe some special features enjoyed by SSS- pseudo linear functionals on a SSS- pseudo linear algebra over  $R = [0, n)$ .
13. Obtain Bassel's inequality for SSS-inner product spaces.
14. Give any other special feature associated with SSS-inner product space.

15. Give some special properties enjoyed by SSS-linear functionals on  $V$ ;  $V$  a SSS- pseudo vector space over a ring  $[0, n)$   $n < \infty$ .

16. Let  $V = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in [0, 22) \right\}$  be a SSS- pseudo linear

algebra over the S-special pseudo interval ring  $R=[0, 22)$ .

- (i) What is dimension of  $V$ , a SSS- pseudo vector space over  $R = [0, 22)$ ?
- (ii) What is the dimension of  $V$  as a SSS- pseudo linear algebra over  $R = [0, 22)$ ?
- (iii) Show  $V$  can have SSS-quasi pseudo vector subspaces over  $R = [0, 22)$ .
- (iv) Can a inner product  $\langle \cdot \rangle_{SSS}$  be defined on  $V$ ?
- (v) Can a SSS-linear functional be defined on  $V$ ?

17. Let  $V = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in [0, 29) \right\}$  be a SSS- pseudo linear

algebra over the S-special pseudo interval ring  $R = [0, 29)$ .

Study questions (i) to (v) of problem 16 for this  $V$ .

18. Let  $V = \left\{ \sum_{i=0}^{20} a_i x^i \mid a_i \in [0, 13), 0 \leq i \leq 20 \right\}$  be a SSS-

pseudo linear algebra over the S-special interval pseudo ring  $R = [0, 13)$ .

- (i) Prove  $W$  is only a SSS- pseudo vector space and is not a SSS- pseudo linear algebra over  $R = [0, 13)$ .
- (ii) Find a basis of  $W$  over  $R$ .
- (iii) Is  $W$  finite dimensional?
- (iv) Can  $W$  have SSS-vector subspaces?
- (v) Can a SSS-inner product be defined on  $W$ ?

(vi) Find  $W^* = \{\text{Collection of all SSS-linear functions on } W\}$ .

(vii) Can SSS-linear operators be defined on  $W$ ?

19. Let  $M = \left\{ \sum_{i=0}^{12} a_i x^i \mid a_i \in [0, 11), 0 \leq i \leq 12 \right\}$  be a SSS-linear pseudo algebra over the S-special interval pseudo ring  $R = [0, 11)$ .

Study questions (i) to (vii) of problem 18 for this  $M$ .

20. Let  $N = \left\{ \sum_{i=0}^7 a_i x^i \mid a_i \in [0, 5), 0 \leq i \leq 7 \right\}$  be a SSS-linear pseudo algebra over the S-special pseudo interval ring  $R = [0, 5)$ .

Study questions (i) to (vii) of problem 18 for this  $N$ .

21. Let  $V = \{(a_1, a_2, \dots, a_{12}) \mid a_i \in [0, 11), 1 \leq i \leq 12\}$  and

$$W = \left\{ \begin{bmatrix} a_1 & a_2 \\ \vdots & \vdots \\ a_{11} & a_{12} \end{bmatrix} \mid a_i \in [0, 11); 1 \leq i \leq 12 \right\} \text{ be two}$$

SSS-vector space over the S-special pseudo interval ring  $R = [0, 11)$ .

- (i) Find  $\text{Hom}_R(V, W)$ .
- (ii) What is the algebraic structure enjoyed by  $\text{Hom}_R(V, W)$ ?
- (iii) Is  $\text{Hom}_R(V, W)$  a SSS-pseudo vector space over  $R$ ?
- (iv) Find  $H_R(V, V)$  and  $\text{Hom}_R(W, W)$ .
- (v) Is  $H_R(V, V) \cong \text{Hom}_R(W, W)$ ?



- (vi) Is  $\text{Hom}_R(V, V)$  and  $\text{Hom}_R(W, W)$  SSS- pseudo linear algebra over  $R$  of same finite dimension.

$$22. \quad \text{Let } V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & \dots & \dots & \dots & a_{12} \\ a_{13} & a_{14} & \dots & \dots & \dots & a_{18} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{31} & a_{32} & \dots & \dots & \dots & a_{36} \end{bmatrix} \right\} \quad a_i \in [0, 19);$$

$1 \leq i \leq 36\}$  be the SSS- pseudo linear algebra under  $\times$  and

$$W = \left\{ \begin{pmatrix} a_1 & \dots & a_{18} \\ a_{19} & \dots & a_{36} \end{pmatrix} \right\} \quad a_i \in [0, 19); 1 \leq i \leq 36\}$$

be SSS- pseudo linear algebra.

Study questions (i) to (vi) of problem 21 for this  $V$  and  $W$ .

$$23. \quad \text{Let } V = \left\{ \left( \begin{array}{ccc|ccc} a_1 & a_2 & & a_3 & a_4 & a_5 & & a_6 \\ a_7 & a_8 & & a_9 & a_{10} & a_{11} & & a_{12} \end{array} \right) \right\} \quad a_i \in [0, 14);$$

$1 \leq i \leq 12, +, \times_n\}$  be the SSS- pseudo linear algebra over the S-special interval pseudo ring  $R = [0, 14)$ .

Study questions (i) to (vii) of problem 18 for this  $V$ .

$$24. \quad \text{Let } V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{bmatrix} \right\} \quad a_i \in [0, 39); 1 \leq i \leq 12\}$$

SSS- pseudo linear algebra over the S-special pseudo interval ring  $R = [0, 39)$ .

(i) Define three distinct pseudo inner products on  $V$ .

(ii) Find  $W^\perp$  given  $W = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ 0 \end{bmatrix} \mid a_1, a_2 \in [0, 39) \right\} \subseteq V$

is a SSS- pseudo subspace of  $V$ .

25. Obtain some special and interesting features enjoyed by pseudo inner product on SSS- pseudo vector space over the S-special pseudo interval ring  $R = [0, n)$ .

26. Let  $M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & \dots & \dots & \dots & a_{10} \\ \hline a_{11} & \dots & \dots & \dots & a_{15} \\ a_{16} & \dots & \dots & \dots & a_{20} \\ \hline a_{21} & \dots & \dots & \dots & a_{25} \end{bmatrix} \mid a_i \in [0, 28); \right.$

$1 \leq i \leq 25\}$  be a SSS-pseudo vector space over the S-special pseudo interval ring  $R = [0, 28)$ .

(i) Define a pseudo inner product on  $V$ .

(ii) Define  $f_{\text{sss}} : V \rightarrow [0, 28)$  and find  $V_{\text{sss}}^*$ .

(iii) Find a basis of  $V$  and  $V_{\text{sss}}^*$  ( $V_{\text{sss}}^*$  is the SSS-dual pseudo vector space of  $V$  the SSS-vector space).

$$27. \quad \text{Let } V = \left\{ \left[ \begin{array}{ccc|ccc|ccc} a_1 & a_2 & a_3 & & & & & & \\ a_4 & a_5 & a_6 & & & & & & \\ a_7 & a_8 & a_9 & & & & & & \\ \hline a_{10} & a_{11} & a_{12} & & & & & & \\ a_{13} & a_{14} & a_{15} & & & & & & \\ a_{16} & a_{17} & a_{18} & & & & & & \\ \hline a_{19} & a_{20} & a_{21} & & & & & & \\ a_{22} & a_{23} & a_{24} & & & & & & \\ a_{25} & a_{26} & a_{27} & & & & & & \\ a_{28} & a_{29} & a_{30} & & & & & & \end{array} \right] \mid a_i \in [0, 17); 1 \leq i \leq 30 \right\}$$

be the SSS- pseudo vector space over the S-special pseudo interval ring  $R = [0, 17)$ .

Study questions (i) to (iii) of problem 26 for this  $V$ .

$$28. \quad \text{Let } V = \left\{ \left[ \begin{array}{cccc|cccc|cccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & & & & \\ a_9 & \dots & \dots & \dots & \dots & \dots & \dots & a_{16} & & & & \\ a_{17} & \dots & \dots & \dots & \dots & \dots & \dots & a_{24} & & & & \\ a_{25} & \dots & \dots & \dots & \dots & \dots & \dots & a_{32} & & & & \end{array} \right] \mid a_i \in$$

$[0, 26); 1 \leq i \leq 32 \}$  be the SSS- pseudo vector space over the S-special interval pseudo ring  $R = [0, 26)$ .

Study questions (i) to (iii) of problem 26 for this  $V$ .

$$29. \quad \text{Let } V = \{(a_1, a_2, a_3, a_4 \mid a_5 \ a_6 \ a_7 \mid a_8 \ a_9 \mid a_{10}) \mid a_i \in [0, 86); 1 \leq i \leq 10\}$$

be the SSS- pseudo vector space over the S-special interval pseudo ring  $R = [0, 86)$ .

Study questions (i) to (iii) of problem 26 for this  $V$ .

30. Let  $V = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in [0, 23) \right\}$  be the SSS- pseudo vector space over the S- pseudo special interval ring  $R = [0, 23)$ .

- (i) Give some special properties enjoyed by this  $V$ .
- (ii) Can  $V$  have finite dimensional SSS-pseudo vector subspaces?
- (iii) Can  $V$  have infinite dimensional SSS-pseudo vector subspaces?
- (iv) Give a basis of  $V$ .
- (v) How many basis can  $V$  have?

31. Let  $V = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in [0, 48) \right\}$  be a SSS- pseudo linear algebra over the S- pseudo special interval ring  $R = [0, 48)$ .

Study questions (i) to (v) of problem 30 for this  $V$ .

32. Let  $V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \mid a_i \in [0, 19); 1 \leq i \leq 5 \right\}$  be the SSS-

pseudo vector space over the S- pseudo special interval ring  $R = [0, 19)$ .

- (i) Find  $V^*$  of  $V$ .
- (ii) What is the dimension of  $V^*$ ?

$$(iii) \quad \text{If } V = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 11 \end{bmatrix} \right\} \text{ is a basis of } V \text{ find}$$

the corresponding basis for  $V^*$ .

(iv) Find a basis of  $\text{Hom}_R(V, V)$  over  $R$ .

(v) Define an inner product on  $V$ .

$$33. \quad \text{Let } V = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_9 \\ a_{10} & a_{11} & \dots & a_{18} \\ a_{19} & a_{20} & \dots & a_{27} \\ a_{28} & a_{29} & \dots & a_{36} \\ a_{37} & a_{38} & \dots & a_{45} \\ a_{46} & a_{47} & \dots & a_{54} \end{bmatrix} \mid a_i \in [0, 43); 1 \leq i \leq 54 \right\}$$

be a SSS- pseudo vector space over the S-special pseudo interval ring  $R = [0, 43)$ .

Study questions (i) to (v) of problem 32 for this  $V$ .

$$34. \quad \text{Let } V = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_{12} \\ a_{13} & a_{14} & \dots & a_{24} \\ a_{25} & a_{26} & \dots & a_{36} \end{bmatrix} \mid a_i \in [0, 53); 1 \leq i \leq 36 \right\}$$

be a SSS- pseudo vector space over the S-special pseudo interval ring  $R = [0, 53)$ .

Study questions (i) to (v) of problem 32 for this  $V$ .

35. Let  $V = \left\{ \sum_{i=0}^{25} a_i x^i \mid a_i \in [0, 41), 0 \leq i \leq 25 \right\}$  be a SSS-

pseudo linear algebra over the S- pseudo special interval ring  $R = [0, 41)$ .

- (i) Study questions (i) to (v) of problem 32 for this  $V$ .
- (ii) Show  $V$  is finite dimensional.
- (iii) Find a basis of  $V$  over  $R$ .

36. Let  $V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_7 \end{bmatrix} \mid a_i \in \left\{ \sum_{i=0}^{\infty} g_i x^i \mid g_i \in [0, 46) \right\} \right\}$  be the SSS-

pseudo vector space over the S- pseudo special interval ring  $R = [0, 46)$ .

Study questions (i) to (v) of problem 32 for this  $S$ .

37. Let  $V = \{(a_1, a_2, a_3, \dots, a_9) \mid a_i \in \left\{ \sum_{i=0}^{\infty} g_i x^i \mid g_i \in [0, 29), \right.$

$1 \leq i \leq 9\}$  be the SSS- pseudo vector space over the S- pseudo special interval ring  $R = [0, 29)$ .

- (i) Prove  $V$  is also a SSS- pseudo linear algebra.
- (ii) What is the dimension of  $V$  as a SSS- pseudo vector space over  $R$ ?
- (iii) What is the dimension of  $V$  as a SSS- pseudo linear algebra over  $R$ ?
- (iv) Find SSS- pseudo sublinear algebras.
- (v) Prove  $V$  has SSS-quasi vector spaces of finite dimension over  $R$ .

$$38. \quad \text{Let } V = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{30} \\ a_{31} & a_{32} & \dots & a_{40} \end{bmatrix} \mid a_i \in \left\{ \sum_{i=0}^{\infty} g_i x^i \mid g_i \in \right.$$

$[0, 93), 1 \leq i \leq 40\}$  be the SSS- pseudo linear algebra under natural product  $\times_n$  over the S- pseudo special interval ring  $R = [0, 93)$ .

Study questions (i) to (v) of problem 37 for this S.

$$39. \quad \text{Let } V = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_{16} \\ a_{17} & a_{18} & \dots & a_{32} \\ a_{33} & a_{34} & \dots & a_{48} \\ a_{49} & a_{50} & \dots & a_{64} \end{bmatrix} \mid a_i \in \left\{ \sum_{i=0}^{\infty} g_i x^i \mid g_j \in \right.$$

$[0, 6), 1 \leq i \leq 64\}$  be the SSS- pseudo linear algebra under natural product  $\times_n$  over  $R = [0, 6)$ .

Study questions (i) to (v) of problem 37 for this M.

$$40. \quad \text{Let } V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix} \mid a_i \in \left\{ \sum_{i=0}^{10} g_i x^i \mid g_i \in [0, 41), 1 \leq i \leq 9 \right\} \right\}$$

the SSS-vector space over the S-pseudo special interval ring  $R = [0, 41)$ .

- (i) Prove  $V$  is not a SSS- pseudo linear algebra.
- (ii) Is  $V$  finite dimensional?

- (iii) Find a basis of  $V$  over  $R$ .
- (iv) Find  $\text{Hom}_R(V, V)$ .
- (v) Find  $V^*$  of  $V$ .
- (vi) Define a pseudo inner product on  $V$ .

$$(vii) \text{ Find for the set } A = \left\{ \begin{bmatrix} 9x+2 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\} \subseteq V. \quad A^\perp \text{ is } A^\perp a$$

SSS- pseudo subspace of  $V$  over  $R$ .

$$41. \text{ Let } M = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_9 \\ a_{10} & a_{11} & \dots & a_{18} \\ a_{19} & a_{20} & \dots & a_{27} \end{bmatrix} \mid a_i \in \left\{ \sum_{i=0}^{12} g_i x^i \mid g_j \in [0, \right.$$

$5), 0 \leq j \leq 12, 1 \leq i \leq 27 \}$  be the SSS- pseudo vector space over the S- pseudo special interval ring  $R = [0, 5)$ .

Study questions (i) to (vii) of problem 40 for this  $M$ .

$$42. \text{ Let } T = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix} \mid a_i \in \left\{ \sum_{i=0}^{16} g_i x^i \mid g_j \in [0, 15), \right.$$

$0 \leq j \leq 16, 1 \leq i \leq 30 \}$  be the SSS- pseudo vector space over the S-special pseudo interval ring  $R = [0, 15)$ .

Study questions (i) to (vii) of problem 40 for this  $T$ .



43. Let  $V = \{(a_1 \mid a_2 \ a_3 \mid a_4 \ a_5 \ a_6 \mid a_7 \ a_8 \ a_9 \ a_{10} \mid a_{11} \ a_{12} \ a_{13} \ a_{14} \ a_{15}$

$$a_{16}) \mid a_i \in \left\{ \sum_{i=0}^5 g_i x^i \mid g_j \in [0, 7), 0 \leq j \leq 5, 1 \leq i \leq 16 \right\} \text{ be}$$

the SSS-vector space over the S-special pseudo interval ring  $R = [0, 7)$ .

Study questions (i) to (vii) of problem 40 for this  $V$ .

44. Let  $V = \left\{ \left[ \begin{array}{c|ccc|cc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_8 & \dots & \dots & \dots & \dots & \dots & a_{14} \\ a_{15} & \dots & \dots & \dots & \dots & \dots & a_{21} \\ \hline a_{22} & \dots & \dots & \dots & \dots & \dots & a_{28} \\ a_{29} & \dots & \dots & \dots & \dots & \dots & a_{35} \\ \hline a_{36} & \dots & \dots & \dots & \dots & \dots & a_{42} \\ a_{43} & \dots & \dots & \dots & \dots & \dots & a_{49} \\ \hline a_5 & \dots & \dots & \dots & \dots & \dots & a_{56} \end{array} \right] \mid a_i \in$

$$\left\{ \sum_{i=0}^3 g_i x^i \mid g_j \in [0, 13), 0 \leq j \leq 3, 1 \leq i \leq 56 \right\} \text{ be the SSS-}$$

pseudo vector space over the S-special interval pseudo ring  $R = [0, 13)$ .

Study questions (i) to (vii) of problem 40 for this  $V$ .

$$45. \quad \text{Let } V = \left[ \begin{array}{ccc} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ \hline a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ \hline a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} \\ \hline a_{22} & a_{23} & a_{24} \\ a_{25} & a_{26} & a_{27} \\ \hline a_{28} & a_{29} & a_{30} \\ \hline a_{31} & a_{32} & a_{33} \\ \hline a_{34} & a_{35} & a_{36} \end{array} \right] \quad a_i \in \left\{ \sum_{i=0}^2 g_i x^i \mid g_j \in [0, 2), \right.$$

$0 \leq j \leq 7, 1 \leq i \leq 36\}$  be the SSS-pseudo vector space over the S-special pseudo interval ring  $R = [0, 2)$ .

- (i) Study questions (i) to (vii) of problem 40 for this  $V$ .
- (ii) If we put  $x^8 = 1$  can  $V$  be made into a SSS- pseudo linear algebra under the natural product  $\times_n$  of matrices.
- (iii) Find dimension of  $V$  as a SSS-pseudo linear algebra over  $R$ .

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On India's 60th Independence Day, Dr. Vasantha was conferred the Kalpana Chawla Award for Courage and Daring Enterprise by the State Government of Tamil Nadu in recognition of her sustained fight for social justice in the Indian Institute of Technology (IIT) Madras and for her contribution to mathematics. The award, instituted in the memory of Indian-American astronaut Kalpana Chawla who died aboard Space Shuttle Columbia, carried a cash prize of five lakh rupees (the highest prize-money for any Indian award) and a gold medal.

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**In this book a new type of linear algebras called special pseudo linear algebras using the intervals  $[0, n]$  is defined. Several of their properties are analysed and some open problems are proposed in this book.**

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